

AUTOMORPHISM GROUPS OF UNIFORM DESSINS D'ENFANTS OF GENUS AT LEAST TWO

TATSUYA OHNISHI

ABSTRACT. For a smooth projective algebraic curve defined over a number field, one can associate a bipartite graph called a *dessin d'enfant*.

We study the regularity and automorphism groups of dessins with *uniform passports*. In a previous paper, we proved that every passport of the form $[n, b^q, n]$ of genus at least 2 admits a dessin with trivial automorphism group. Here we prove the analogous result for passports of the form $[b^q, b^q, n]$.

We also construct examples of uniform passports of genus at least 2 for which every dessin with that passport has nontrivial automorphism group, and others for which every dessin with that passport has trivial automorphism group.

Finally, we give an alternative proof of the $[n, b^q, n]$ case using counting arguments based on centralizers of permutations.

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Tatsuya Ohnishi (大西 達也)

Graduate School of Information Science and Technology, The University of Osaka, Japan

e-mail: ohnishi-t@ist.osaka-u.ac.jp

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1. Introduction

1.1. Belyĭ's Theorem and Dessins d'Enfants

Theorem 1.1 (Belyĭ's Theorem). [Bel79][Bel02][JW16, Theorem 1.3] *Let X be a compact Riemann surface, that is, a smooth projective algebraic curve in $\mathbb{P}_{\mathbb{C}}^N$ for some N . Then X can be defined over the field of algebraic numbers $\overline{\mathbb{Q}}$ if and only if there exists a nonconstant meromorphic function $\beta: X \rightarrow \widehat{\mathbb{C}}$ ($:= \mathbb{C} \cup \{\infty\}$) ramified over at most three points.*

This deep theorem reveals a surprising connection between algebraic curves over number fields and combinatorial structures on surfaces. This connection deeply fascinated Grothendieck and led to his theory of dessins d'enfants.

For any compact Riemann surface X defined over a number field, one can choose a suitable function β (called a *Belyĭ function*) such that all of its critical values lie among three points. By applying an appropriate Möbius transformation, these points may be taken to be 0, 1, and ∞ .

For such a pair (X, β) (called a *Belyĭ pair*), one can associate a bipartite graph called a *dessin d'enfant* (child's drawing), or simply a *dessin* (see Definition 2.1).

The dessin for (X, β) is drawn on an orientable surface of the same genus as X , where the black vertices ($\bullet$) and white vertices ($\circ$) represent $\beta^{-1}(0)$ and $\beta^{-1}(1)$, respectively¹. The embedded graph is $\beta^{-1}([0, 1])$. Its edges are the closures of the connected components of $\beta^{-1}((0, 1))$, and its faces are the connected components of $X \setminus \beta^{-1}([0, 1])$. Each face is homeomorphic to an open disk and contains exactly one point of $\beta^{-1}(\infty)$.

By studying the properties of dessins d'enfants, one can combine insights from algebraic geometry and combinatorics, including graph theory, to enrich both subjects. This perspective also opens up a range of possibilities for further applications.

Two dessins $\mathcal{D}$ and $\mathcal{D}'$ are said to be *isomorphic* if there exists an orientation-preserving homeomorphism between the underlying surfaces that induces an isomorphism of the embedded bipartite graphs, preserving vertex colors and the cyclic order of incident edges at each vertex. Equivalently, if (X, β) and (X', β') are the corresponding Belyĭ pairs, then $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic if and only if there exists a biholomorphic map $\varphi: X \rightarrow X'$ such that $\beta' \circ \varphi = \beta$.

Two examples of dessins d'enfants are shown in Figure 1. Since $X = \widehat{\mathbb{C}}$ has genus 0 in both cases, the corresponding dessins can be drawn on a sphere. For the left dessin, we may take

$$\beta(z) = -\frac{5z^3(3z - 6 + \sqrt{6})^2}{(9 - 4\sqrt{6})(-15z + 12 + 2\sqrt{6})},$$

as a Belyĭ function. The number of edges of a dessin is equal to the degree of the corresponding Belyĭ function, which is 5 in this case.

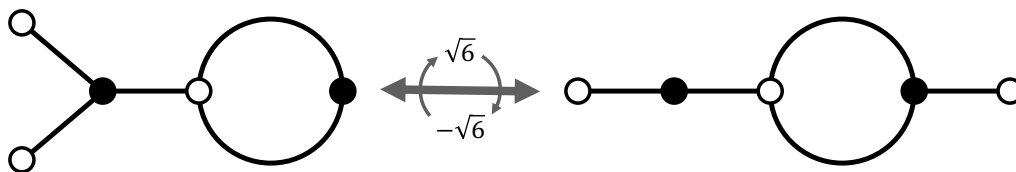


FIGURE 1. A pair of Galois-conjugate dessins d'enfants

¹ Some references (such as [JW16]) represent $\beta^{-1}(0)$ by white vertices and $\beta^{-1}(1)$ by black vertices. In this paper, we adopt the convention used in many classical references.

By applying the Galois automorphism $\sqrt{6} \mapsto -\sqrt{6}$ to the coefficients of $\beta(z)$, we obtain the dessin on the right. This is an example of an action of the absolute Galois group $\mathbb{G} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins d'enfants [JW16, 4.2.1].

The action of $\mathbb{G}$ on the set of all dessins is faithful; that is, for any two distinct elements of $\mathbb{G}$, there exists a dessin whose images under these elements lie in distinct isomorphism classes. Therefore, the study of Galois orbits of dessins provides a powerful tool for investigating the structure of the absolute Galois group.

1.2. Regularity and Automorphism Groups

Regularity and the isomorphism class of the automorphism group are invariant under the Galois action. A dessin is said to be *regular* if its *monodromy group* (see Definition 2.4) acts regularly (that is, freely and transitively) on the set of edges. From a geometric point of view, the *automorphism group* of a dessin is the group of deck transformations of the associated Belyĭ covering. Equivalently, it can be identified with the centralizer of the monodromy group in the symmetric group acting on the edges.

By studying regularity and automorphism groups, one can gain insight into the symmetry properties of a dessin d'enfant. The order of the automorphism group always divides the number of edges of the dessin, and the equality of these two numbers is equivalent to the dessin being regular. Thus, the structure of the automorphism group provides a precise measure of the symmetry exhibited by the dessin.

Figure 2 shows two dessins of genus 1 with the same passport; that is, they have the same valency list: each has 8 edges, two black vertices of valency 4, four white vertices of valency 2, and two faces of valency 4. Although both dessins may appear highly symmetric at first glance, the left dessin is regular, whereas the right one is not. The automorphism group of the left dessin has order 8, while that of the right dessin has order 4.

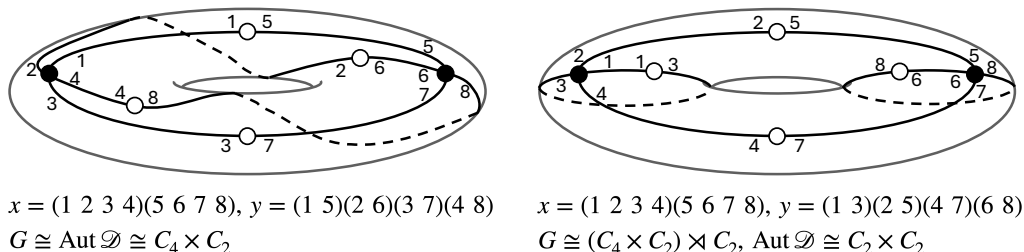


FIGURE 2. Regular and nonregular dessins with the same passport

Thus, regularity and automorphism groups may differ among dessins with the same passport. However, they are invariant under the action of the absolute Galois group. They therefore play an essential role in the study of families of dessins lying in the same Galois orbit.

Furthermore, regularity and automorphism groups have a significant impact on the relationship between the field of moduli and the field of definition.

For a dessin d'enfant $\mathcal{D}$ corresponding to a Belyĭ pair (X, β) , the *field of moduli* $M(\mathcal{D})$ is defined as the fixed field of the subgroup $G(\mathcal{D}) = \{\sigma \in \mathbb{G} \mid \mathcal{D} \cong \mathcal{D}^\sigma\} \leq \mathbb{G}$. That is, $M(\mathcal{D})$ is the subfield of $\overline{\mathbb{Q}}$ consisting of those elements that are fixed by every automorphism σ for which the conjugate dessin $\mathcal{D}^\sigma$ is isomorphic to $\mathcal{D}$. For the dessins in Figure 1, $M(\mathcal{D})$ is $\mathbb{Q}(\sqrt{6})$.

A number field K is called a *field of definition* of $\mathcal{D}$ if both X and β can be defined over K . Unlike the field of moduli, a field of definition of $\mathcal{D}$ is in general not unique, and there need not exist a smallest field of definition.

The field of moduli depends only on the isomorphism class of $\mathcal{D}$ and is contained in any field of definition of $\mathcal{D}$. However, the field of moduli is not always a field of definition. For certain important classes of dessins, including dessins with trivial automorphism groups and regular dessins, the field of moduli is a field of definition [SV16][CJSW13].

In this paper, we develop methods for analyzing how regular dessins, dessins with trivial automorphism groups, and nonregular dessins with nontrivial automorphism groups are distributed. The facts above provide an additional motivation for this study.

1.3. Uniform Passports and Dessins

A dessin d'enfant is said to be *uniform* if the valencies of black vertices, white vertices, and faces are each constant. One also says that it has a uniform passport (or uniform valency list).

Uniformity expresses a local combinatorial regularity: the valencies of the black vertices, white vertices, and faces are constant within each class. Every regular dessin is uniform, but the converse does not hold. Uniformity alone does not imply the existence of any nontrivial global symmetry; a uniform dessin may even have a trivial automorphism group. In contrast, regularity is a global symmetry condition.

As an example, consider uniform passports $[a^p, b^q, c^r]$, where $n = pa = bq = rc$, and suppose that $n = 6$. By symmetry among black vertices, white vertices, and faces, we may assume $c \geq a \geq b$ (equivalently, $r \leq p \leq q$).

In genus 0, the uniform passports are $[6, 1^6, 6]$ and $[2^3, 2^3, 3^2]$. The corresponding dessins are shown in Figure 3. In both cases, the dessins are regular.

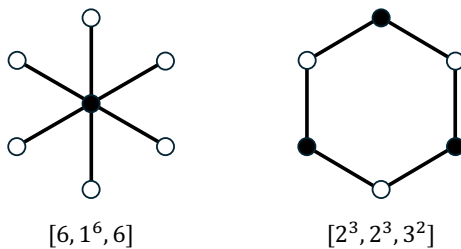


FIGURE 3. Uniform dessins d'enfants of genus 0, degree 6

In genus 1, the uniform passports are $[3^2, 2^3, 6]$ and $[3^2, 3^2, 3^2]$, and the corresponding dessins are shown in Figure 4. The dessin on the left is regular, whereas the one on the right is not. The automorphism group of the right-hand dessin has order 2, which is strictly smaller than 6.

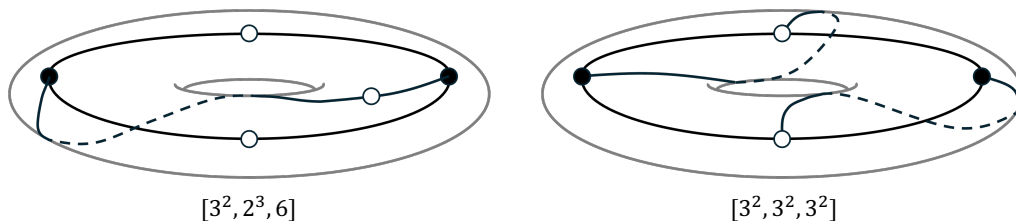


FIGURE 4. Uniform dessins d'enfants of genus 1, degree 6

For genus at least 2, the only uniform passport is $[6, 3^2, 6]$, which has genus 2. There are four dessins with this passport, shown in Figure 5. The orders of their automorphism groups are 6, 3, 2, and 1, respectively, from the upper left to the lower right. Only the upper-left dessin is regular; the others are nonregular, and the lower-right dessin has a trivial automorphism group.

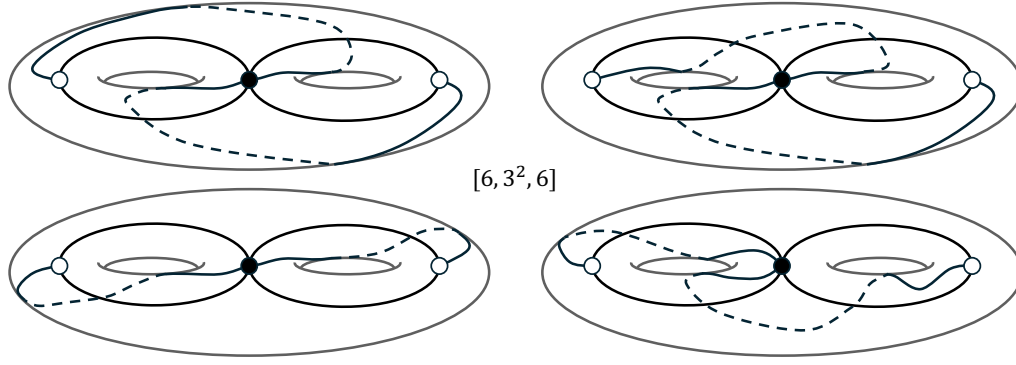


FIGURE 5. Uniform dessins d'enfants of genus 2, degree 6

1.4. Previous Work

Examination of the regularity and automorphism groups of uniform dessins d'enfants suggests that, as the genus increases, the proportion of regular dessins tends to decrease, whereas the proportion of dessins with trivial automorphism groups tends to increase.

For a given uniform passport, the following questions naturally arise:

- Under what conditions do regular dessins d'enfants exist, and how many are there?
- How are the automorphism groups distributed?
- How does this distribution change with the genus?

To the best of our knowledge, relatively little attention has been paid to the study of uniform passports as families of dessins and to the systematic investigation of their regularity and automorphism groups. In our previous paper [Ohn26a], we proposed the conjectural picture shown in Table 1 and established several supporting results.

Genus	$\text{Aut } \mathcal{D} \cong \{1\}$	$1 < \text{Aut } \mathcal{D} < n$	$ \text{Aut } \mathcal{D} = n$ (regular)
0	– (1)	– (1)	✓ ⁽¹⁾
1 (with an (n) -cycle)	– (3)	– (2)	✓ ⁽⁵⁾
1 (without an (n) -cycle)	– (3)	✓ ⁽²⁾	◇
≥ 2 (with ≥ 2 (n) -cycles)	✓ ⁽⁴⁾	◇	✓ ⁽⁵⁾
≥ 2 (with ≤ 1 (n) -cycle)	✓*	✓*	◇

n : number of edges of the dessin d'enfant

✓: always occurs, –: never occurs, ◇: depends on the passport.

In this table, we assume that $n \geq 2$. If $n = 1$ is allowed, the unique dessin with passport $[1, 1, 1]$ belongs to both the $\text{Aut } \mathcal{D} \cong \{1\}$ column and the $|\text{Aut } \mathcal{D}| = n$ (regular) column.

TABLE 1. Distribution of automorphism groups for uniform passports (Previous)

Asterisks (*) indicate statements that were conjectural in the initial version of [Ohn26a]. The numbers in parentheses refer to the corresponding items listed below.

- (1) Genus 0: all uniform dessins are regular and have automorphism groups of order n , isomorphic to their monodromy groups.
- (2) Genus 1: a uniform passport admits a nonregular dessin if and only if none of its three components is (n) .
- (3) Genus 1: no uniform passport admits a dessin with a trivial automorphism group.
- (4) Genus ≥ 2 : original conjecture: every uniform passport admits a dessin with a trivial automorphism group, and hence admits a nonregular dessin. Moreover, if

the passport has at most one entry equal to (n) , then it also admits a nonregular dessin with a nontrivial automorphism group.

- (5) Any genus: a passport of the form $[a^p, b^q, n]$, or any permutation thereof, admits a regular dessin if and only if $\gcd(p, q) = 1$.

In [Ohn26a] we proved (1), (2), (3), and (5), and partially proved (4) for the case of $[n, b^q, n]$ with $q \geq 1$.

These results clarify several aspects of the regularity and automorphism groups of uniform dessins. They describe the distribution of automorphism groups in genera 0 and 1, establish a necessary and sufficient condition for a passport of the form $[a^p, b^q, n]$, or any permutation thereof, to admit a regular dessin, and provide a group-theoretic characterization of regularity for general uniform passports. For passports $[n, b^q, n]$ of genus at least 2, we developed a method for proving the existence of dessins with trivial automorphism groups by estimating the number of relevant permutations and analyzing the primitivity of the monodromy group.

1.5. Main Results

In this paper, we provide the following results.

- (a) Extending the result corresponding to (4) in Table 1, we prove the following theorem in Section 4.

Theorem 4.7. *If a uniform passport $[b^q, b^q, n]$ has genus at least 2, then it admits a dessin d'enfant with a trivial automorphism group.*

The same statement holds for the passports $[b^q, n, b^q]$ and $[n, b^q, b^q]$.

For the conjectural parts of Table 1, we found several counterexamples. We discuss the following in Section 5.

- (b) The passport $[8^2, 2^8, 4^4]$ admits no dessin d'enfant with trivial automorphism group. This provides a counterexample to the corresponding conjecture in Table 1 and reveals a new family of passports whose automorphism groups exhibit unexpected behavior.
- (c) Conversely, the passport $[5^3, 5^3, 15]$ admits no dessin d'enfant with a nontrivial automorphism group. This provides a counterexample to the corresponding conjecture in Table 1. Generalizing this observation, we prove the following theorem.

Theorem 5.1. *Let ℓ_1 and ℓ_2 be distinct prime numbers with $\ell_1 < \ell_2$. Then every dessin d'enfant with passport $[\ell_2^{\ell_1}, \ell_2^{\ell_1}, \ell_1 \ell_2]$ has trivial automorphism group.*

The same conclusion holds for the passports $[\ell_1 \ell_2, \ell_2^{\ell_1}, \ell_2^{\ell_1}]$ and $[\ell_2^{\ell_1}, \ell_1 \ell_2, \ell_2^{\ell_1}]$.

- (d) For passports of the form $[n, n, n]$ of genus at least 2, we prove the following theorem in Section 6.

Theorem 6.1. *Let $[n, n, n]$ be a uniform passport of genus at least 2. Then it admits a nonregular dessin d'enfant with nontrivial automorphism group if and only if n is composite.*

Furthermore, if n is composite, then for every divisor r of n satisfying $1 < r < n$, there exists a dessin $\mathcal{D}$ with passport $[n, n, n]$ such that $|\text{Aut } \mathcal{D}| = r$.

Remark 1.2. [Hor24] gives a method for computing the number of isomorphism classes of dessins with two vertices (one black and one white) and L faces whose automorphism group has order r , where r is a divisor of the number of edges. The passport $[n, n, n]$ corresponds to the case $L = 1$.

In this paper, we provide an explicit construction of a dessin whose automorphism group has order r .

Moreover,

- (e) In Section 7, using the centralizer estimates developed in Section 3, we give an alternative proof of [Ohn26a, Theorem 7.6], which states that every uniform passport $[n, b^q, n]$ of genus at least 2 admits a dessin d'enfant with a trivial automorphism group.

Table 2 summarizes the results of this paper and the conjectures for uniform passports of genus at least 2 remaining at the time of its original version. Asterisks (*) indicate statements that were conjectural at that time. The letters in parentheses indicate the corresponding items listed above.

Passport	$\text{Aut } \mathcal{D} \cong \{1\}$	$1 < \text{Aut } \mathcal{D} < n$	$ \text{Aut } \mathcal{D} = n$ (regular)
$[n, n, n]$ (n : prime)	✓	— ^(d)	✓
$[n, n, n]$ (n : composite)	✓	✓ ^(d)	✓
$[n, b^q, n]$ ($q \geq 2$)	✓ ^(e)	✓*	✓
$[b^q, b^q, n]$ ($q \geq 2$)	✓ ^(a)	◇ ^(c)	—
$[a^p, b^q, n]$ ($q > p \geq 2$)	✓*	✓*	✓ ($\gcd(p, q) = 1$) — ($\gcd(p, q) \neq 1$)
$[a^p, b^q, c^r]$ ($p, q, r \geq 2$)	◇ ^(b)	✓*	◇

n : number of edges of the dessin d'enfant

✓: always occurs, —: never occurs, ◇: depends on the passport.

TABLE 2. Distribution of automorphism groups for uniform passports (Genus ≥ 2)

In light of the results obtained in our subsequent work, two directions remain for future research. First, although the existence of a dessin with trivial automorphism group has now been established for every uniform unicellular passport of genus at least 2 [Ohn26b], it remains to characterize when a uniform nonunicellular passport $[a^p, b^q, c^r]$ with $p, q, r \geq 2$ admits such a dessin. It would also be desirable to determine the conditions under which a uniform passport admits a nonregular dessin with nontrivial automorphism group. Second, more quantitative results on the distribution of regular dessins and automorphism groups remain to be obtained.

2. Preliminaries

The definitions and results in this section are based primarily on [JW16] and [APZ20].

Definition 2.1 (Dessin d'enfant). [JW16, Definition 2] A *dessin d'enfant*, or simply a *dessin*, is a map consisting of a connected, finite, bipartite graph embedded in a connected, compact, oriented surface without boundary. Here, a bipartite graph is a graph whose vertices can be colored black and white in such a way that each edge joins a black vertex to a white vertex.

Since a compact Riemann surface provides a suitable surface on which a dessin d'enfant can be embedded, one can draw a dessin corresponding to a Belyĭ pair (X, β) , as described in Section 1.1.

A *partition* λ of a positive integer n , denoted by $\lambda \vdash n$, is a multiset of positive integers whose sum is n , where the order of the parts is irrelevant.

Definition 2.2 (Passport of a dessin). [APZ20, Definition 2.10] Let n be the number of edges of a dessin d'enfant. The triple $[\lambda_0, \lambda_1, \lambda_\infty]$ of partitions $\lambda_0, \lambda_1, \lambda_\infty \vdash n$, which correspond respectively to the valencies of the black vertices, the white vertices, and the faces of the dessin, is called a *passport* of the dessin.

Both dessins in Figure 1 have passport $[32, 311, 41]$, while the dessins in Figure 2 have passport $[4^2, 2^4, 4^2]$. Throughout this paper, we use the standard abbreviated notation for partitions: for example, $32 = (3, 2)$, $311 = (3, 1, 1)$, $41 = (4, 1)$, $4^2 = (4, 4)$, and $2^4 = (2, 2, 2, 2)$.

Since $\lambda_0, \lambda_1, \lambda_\infty$ are partitions of n , we have

$$|\lambda_0| = |\lambda_1| = |\lambda_\infty| = n, \quad (2.1)$$

where $|\lambda|$ denotes the sum of the parts of the partition λ .

The numbers of vertices, faces, and edges are $l(\lambda_0) + l(\lambda_1)$, $l(\lambda_\infty)$, and n , respectively, where $l(\lambda)$ denotes the number of parts of the partition λ . Therefore, Euler's formula gives

$$l(\lambda_0) + l(\lambda_1) + l(\lambda_\infty) - n = 2 - 2g,$$

where g is the genus of the underlying curve X . Hence

$$g = \frac{n - (l(\lambda_0) + l(\lambda_1) + l(\lambda_\infty))}{2} + 1. \quad (2.2)$$

Since g is a nonnegative integer, it follows that

$$l(\lambda_0) + l(\lambda_1) + l(\lambda_\infty) \leq n + 2, \quad l(\lambda_0) + l(\lambda_1) + l(\lambda_\infty) \equiv n \pmod{2}. \quad (2.3)$$

Note that not every triple of partitions satisfying (2.1) and (2.3) is realized by a dessin d'enfant. For example, although $[2^2, 2^2, 31]$ formally satisfies these conditions and would yield genus 0, there exists no dessin with this passport. Similarly, the passport $[3^2, 3^2, 42]$, which would correspond to genus 1, admits no dessin.

These facts can be proved by showing that there exists no corresponding monodromy group (see Definition 2.4) in each case.

Definition 2.3 (Uniform passports and dessins). [JW16, Remark 3.2] The passport of a dessin d'enfant given by $[\lambda_0, \lambda_1, \lambda_\infty] = [a_1 \cdots a_p, b_1 \cdots b_q, c_1 \cdots c_r]$ is called *uniform* if

$$a_1 = \cdots = a_p, \quad b_1 = \cdots = b_q, \quad c_1 = \cdots = c_r,$$

i.e. if the passport takes the form $[a^p, b^q, c^r]$.

A dessin d'enfant is called *uniform* if it has a uniform passport.

For a uniform passport $[a^p, b^q, c^r]$ with $n = pa = qb = rc$, realized by a dessin of genus g , equation (2.2) gives

$$p + q + r - n = 2 - 2g,$$

and hence

$$g = \frac{n - (p + q + r)}{2} + 1. \quad (2.4)$$

Definition 2.4 (Monodromy group of a dessin d'enfant). [JW16, 2.1.1] Define two permutations x and y acting on the set of edges E of a dessin d'enfant $\mathcal{D}$ as follows. For each edge $e \in E$, define $x \cdot e$ and $y \cdot e$ to be the next edges around the unique black vertex

and the unique white vertex incident to e , respectively, following the counterclockwise orientation.

The *monodromy group* of $\mathcal{D}$ is the subgroup $G = \langle x, y \rangle$ generated by x and y in the symmetric group $\text{Sym}(E)$ of all permutations of E .

Since a dessin d'enfant is connected, it follows that any edge in E can be mapped to any other edge by the action of G . Therefore, the monodromy group G acts transitively on E .

Remark 2.5. Throughout this paper, permutations act on the left, and products are composed from right to left, i.e. $(xy) \cdot e = x \cdot (y \cdot e)$ for an edge e .

Another important observation concerning the monodromy group is that, in addition to x and y encoding the cycles of the black and white vertices, respectively, the permutation $z = (xy)^{-1}$ encodes the cycles corresponding to the faces. If a cycle of z has length c , then the corresponding face has valency c in the passport. Equivalently, the boundary of a face of valency c traverses $2c$ edge-sides, counted with multiplicity.

It is known that if a transitive permutation group $G \leq \text{Sym}(E)$ on a finite set E is generated by two elements, then there exists a dessin d'enfant whose monodromy group is G ; this follows from the correspondence between constellations and ramified coverings of the sphere [LZ04, Proposition 1.2.15 and Construction 1.2.17].

Therefore, studying groups that act transitively is essential for investigating the properties of dessins d'enfants and, consequently, of algebraic curves.

Definition 2.6 (Regular dessins). [JW16, 2.1.2] A dessin d'enfant is called *regular* if its monodromy group acts freely (that is, semiregularly) on the set of its edges.

This implies that the monodromy group of a regular dessin acts freely and transitively — hence regularly — on its edges.

The following criterion relates the order of the monodromy group to the number of edges.

Lemma 2.7. *A dessin d'enfant with n edges is regular if and only if the order of its monodromy group is n .*

Proof. See [Ohn26a, Lemma 2.7]. □

Definition 2.8 (Automorphism group of a dessin d'enfant). [JW16, 2.1.2] For a dessin d'enfant $\mathcal{D}$ with edge set E , an *automorphism* of $\mathcal{D}$ is a permutation of E that preserves the cyclic order of edges around each vertex, that is, which commutes with x and y , or equivalently, commutes with G . Thus we can define the *automorphism group* of $\mathcal{D}$ as the centralizer:

$$\begin{aligned} \text{Aut } \mathcal{D} &:= C_{\text{Sym}(E)}(G) = \{c \in \text{Sym}(E) \mid cg = gc \text{ for all } g \in G\} \\ &= \{c \in \text{Sym}(E) \mid cx = xc, cy = yc\}. \end{aligned} \tag{2.5}$$

When $\text{Aut } \mathcal{D} \cong \{1\}$, it is said that $\mathcal{D}$ has a *trivial* automorphism group.

Since $\langle x, y \rangle = \langle x, z \rangle = \langle y, z \rangle$, where $z = (xy)^{-1}$, both the monodromy group and the automorphism group are invariant under permuting the roles of black vertices, white vertices, and faces.

The automorphism group of a dessin d'enfant has the following properties:

- $\text{Aut } \mathcal{D}$ acts freely on the edges of $\mathcal{D}$.
- $|\text{Aut } \mathcal{D}|$ divides the number of edges.
- If $|\text{Aut } \mathcal{D}|$ equals the number of edges, then $\text{Aut } \mathcal{D} \cong G$.

The following proposition is a basic result describing the relationship between regularity and automorphism groups.

Proposition 2.9. *A dessin d'enfant $\mathcal{D}$ is regular if and only if its monodromy group G is isomorphic to $\text{Aut } \mathcal{D}$.*

Proof. See [JW16, Theorem 2.1]. □

This also implies that a dessin d'enfant $\mathcal{D}$ is regular if and only if $|\text{Aut } \mathcal{D}|$ equals the number of edges.

Note that regularity does not imply $G = \text{Aut } \mathcal{D}$; in general these groups act on the edges of the dessin in different ways.

Proposition 2.10. *If a dessin d'enfant is regular, then it has a uniform passport.*

Proof. See [GGD12, Proposition 4.42] and [Ohn26a, Proposition 2.10]. □

The converse of this proposition does not hold in general. Thus, uniformity is necessary but not sufficient for regularity.

Example 2.11. The uniform passport $[4^2, 2^4, 4^2]$ (genus 1) corresponds to the two dessins shown in Section 1.2, Figure 2. The dessin on the left is regular, and both its monodromy group and its automorphism group are isomorphic to $C_4 \times C_2$, the direct product of cyclic groups of orders 4 and 2, respectively. In contrast, the dessin on the right is not regular; its monodromy group is isomorphic to $(C_4 \times C_2) \rtimes C_2$ and has order 16, whereas its automorphism group has order 4 and is isomorphic to $C_2 \times C_2$.

3. Counting Arguments for Trivial Automorphism Groups

3.1. Settings

In this section, we consider uniform passports $[n, b^q, a^p]$ of genus $g \geq 2$ with $n = pa = qb$. That is, we assume that x , y , and $z = (xy)^{-1}$ have cycle types (n) , (b^q) , and (a^p) , respectively.

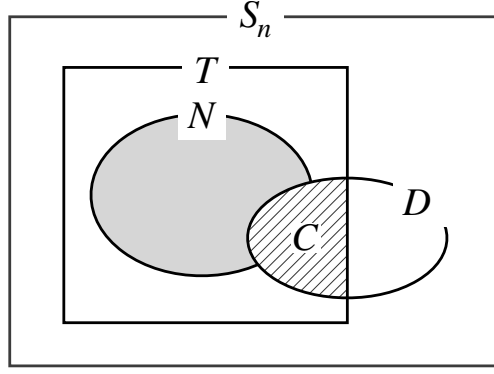
Since the monodromy group and the automorphism group are invariant under permutations of λ_0 , λ_1 , and λ_∞ , it suffices to consider passports of this form; the case $[a^p, b^q, n]$ follows by symmetry.

Let S_n be the symmetric group on $\{1, \dots, n\}$, and fix an n -cycle $x \in S_n$, for example $x = (1 \ 2 \ \dots \ n)$. We define the following subsets of S_n :

$$\begin{aligned}
 T &= T(b, q) := \{y \in S_n \mid y \text{ has cycle type } (b^q)\}, \\
 N &= N(b, q, a) := \{y \in T(b, q) \mid (xy)^{-1} \text{ has cycle type } (a^p)\}, \\
 C &= C(b, q) := \{y \in T(b, q) \mid \text{Aut } \mathcal{D} \not\cong \{1\}\}, \\
 D &= D(n) := \{y \in S_n \mid \text{Aut } \mathcal{D} \not\cong \{1\}\},
 \end{aligned} \tag{3.1}$$

where $\mathcal{D}$ denotes the dessin corresponding to the permutation pair (x, y) , whose monodromy group is $\langle x, y \rangle$.

Figure 6 illustrates the relationships among these sets. The hatched region corresponds to $C(b, q)$.


 FIGURE 6. The sets related to the passport $[n, b^q, a^p]$

The existence of a dessin $\mathcal{D}$ with passport $[n, b^q, a^p]$ and trivial automorphism group is equivalent to

$$N \cap C^c \neq \emptyset$$

(the gray region in Figure 6). Since $C = D \cap T$ and $N \subset T$, it is also equivalent to

$$N \cap D^c \neq \emptyset.$$

Thus, to prove the existence of such a dessin, it suffices to show that

$$|N| > |D|,$$

or equivalently,

$$\frac{|N|}{|T|} > \frac{|D|}{|T|}.$$

Proposition 3.1. *The number of elements in S_n of cycle type (b^q) ($n = bq$) is given by*

$$|T(b, q)| = \frac{n!}{b^q q!}.$$

Proof. See [Ohn26a, Proposition 3.1]. □

3.2. Lower Bound for N

A useful tool for computing the values of $|N|$ is the following theorem from [GS98].

Theorem 3.2. *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_m)$ be partitions of n . Define the genus associated with the pair (λ, μ) by*

$$g = \frac{n - (l + m) + 1}{2},$$

and assume that $g \in \mathbb{Z}_{\geq 0}$.

Let C_λ and C_μ denote the conjugacy classes in S_n consisting of permutations of cycle types λ and μ , respectively. Let $c_{\lambda\mu}^n$ denote the number of solutions $(\sigma, \rho) \in C_\lambda \times C_\mu$ to the equation $\sigma\rho = \pi$, where π is a fixed n -cycle in S_n . Then $c_{\lambda\mu}^n$ is given by

$$c_{\lambda\mu}^n = \frac{n}{z_\lambda z_\mu 2^{2g}} \sum_{\substack{g_1, g_2 \geq 0 \\ g_1 + g_2 = g}} (l + 2g_1 - 1)!(m + 2g_2 - 1)! \sum_{\substack{(i_1, \dots, i_l) \models g_1 \\ (j_1, \dots, j_m) \models g_2}} \prod_{k=1}^l \binom{\lambda_k}{2i_k + 1} \prod_{k=1}^m \binom{\mu_k}{2j_k + 1}.$$

Here $p \models n$ denotes a weak composition of n , that is, a finite sequence of nonnegative integers summing to n , where the order of the terms matters. We adopt the convention that $\binom{a}{b} = 0$ if $b > a$. Moreover, for a partition $\lambda = 1^{\alpha_1} \dots n^{\alpha_n}$, where α_i denotes the multiplicity of i , we define $z_\lambda = \prod_i \alpha_i! i^{\alpha_i}$.

Proof. See [GS98, Theorem 2.1]. □

Using this theorem, we proved the following theorem:

Theorem 3.3. *Let $n \geq 3$ be an integer such that $n = bq$, with $b \geq 2$, and assume that $n \equiv q \pmod{2}$. Fix $x = (1 \ 2 \ \dots \ n) \in S_n$. Then*

$$\frac{|N(b, q, n)|}{|T(b, q)|} \geq \frac{2}{n + 2}.$$

In particular, the inequality becomes an equality when $b = 2$.

Proof. See [Ohn26a, Theorem 3.3]. □

Lemma 3.4. *Let $[n, b^q, b^q]$ be a passport with $n = qb$, and suppose that it has genus at least 1. Fix an n -cycle $x \in S_n$, and let $N(b, q, b)$ denote the set of permutations $y \in S_n$ of cycle type (b^q) such that $(xy)^{-1}$ also has cycle type (b^q) . Then*

$$|N(b, q, b)| \geq \frac{(n - q)!}{2^{q(b-2)} \left(\frac{q+1}{2}\right)! \left(\frac{q-1}{2}\right)!} \left(\frac{b-1}{2b}\right)^{\frac{q-1}{2}}.$$

Proof. Since x^{-1} also has cycle type (n) , Theorem 3.2 implies that $c_{(b^q), (b^q)}^n$ is the number of pairs (σ, ρ) such that both σ and ρ have cycle type (b^q) and satisfy $\sigma\rho = x^{-1}$. Moreover,

$$\sigma\rho = x^{-1} \iff \rho = (x\sigma)^{-1}.$$

Hence $c_{(b^q), (b^q)}^n$ is precisely the number of permutations σ of cycle type (b^q) such that $(x\sigma)^{-1}$ also has cycle type (b^q) , namely $|N(b, q, b)|$.

Therefore, by Theorem 3.2,

$$\begin{aligned} |N(b, q, b)| &= c_{(b^q), (b^q)}^n \\ &= \frac{n}{2^{2g} b^{2q} (q!)^2} \sum_{\substack{g_1, g_2 \geq 0 \\ g_1 + g_2 = g}} (q + 2g_1 - 1)!(q + 2g_2 - 1)! \sum_{\substack{(i_1, \dots, i_q) \models g_1 \\ (j_1, \dots, j_q) \models g_2}} \prod_{k=1}^q \binom{b}{2i_k + 1} \prod_{k=1}^q \binom{b}{2j_k + 1}, \end{aligned}$$

where

$$g = \frac{q(b-2) + 1}{2}.$$

Since $g \geq 1$, the outer sum contains the two distinct terms corresponding to $(g_1, g_2) = (0, g)$ and $(g, 0)$. Hence

$$|N(b, q, b)|$$

$$\geq \frac{n}{2^{2g}b^{2q}(q!)^2} \sum_{\substack{(g_1, g_2)=\\(0, g), (g, 0)}} (q + 2g_1 - 1)!(q + 2g_2 - 1)! \sum_{\substack{(i_1, \dots, i_q) \models g_1 \\ (j_1, \dots, j_q) \models g_2}} \prod_{k=1}^q \binom{b}{2i_k + 1} \prod_{k=1}^q \binom{b}{2j_k + 1}.$$

Since the summand is symmetric in g_1 and g_2 , we obtain

$$|N(b, q, b)|$$

$$\begin{aligned} &\geq 2 \cdot \frac{n}{2^{2g}b^{2q}(q!)^2} \sum_{\substack{(g_1, g_2)=\\(0, g)}} (q + 2g_1 - 1)!(q + 2g_2 - 1)! \sum_{\substack{(i_1, \dots, i_q) \models g_1 \\ (j_1, \dots, j_q) \models g_2}} \prod_{k=1}^q \binom{b}{2i_k + 1} \prod_{k=1}^q \binom{b}{2j_k + 1} \\ &= \frac{qb}{2^{2g-1}b^{2q}(q!)^2} (q - 1)!(q + 2g - 1)! \sum_{\substack{(i_1, \dots, i_q) \models 0 \\ (j_1, \dots, j_q) \models g}} \prod_{k=1}^q \binom{b}{2i_k + 1} \prod_{k=1}^q \binom{b}{2j_k + 1} \\ &= \frac{q!}{2^{2g-1}b^{2q-1}(q!)^2} (q + 2g - 1)! \sum_{(j_1, \dots, j_q) \models g} \prod_{k=1}^q \binom{b}{1} \prod_{k=1}^q \binom{b}{2j_k + 1} \\ &= \frac{1}{2^{2g-1}b^{2q-1}q!} (q + q(b - 2) + 1 - 1)! \sum_{(j_1, \dots, j_q) \models g} b^q \prod_{k=1}^q \binom{b}{2j_k + 1} \\ &= \frac{(n - q)!}{2^{q(b-2)}b^{q-1}q!} \sum_{(j_1, \dots, j_q) \models g} \prod_{k=1}^q \binom{b}{2j_k + 1}. \end{aligned} \tag{3.2}$$

The product $\prod_{k=1}^q \binom{b}{2j_k + 1}$ vanishes whenever $2j_k + 1 > b$ for some k . Hence the sum may be restricted to weak compositions $(j_1, \dots, j_q)$ satisfying $j_k \leq (b - 1)/2$ ($1 \leq k \leq q$). Since

$$\frac{b - 1}{2} \cdot q - g = \frac{q(b - 1)}{2} - \frac{q(b - 2) + 1}{2} = \frac{q - 1}{2},$$

and $q - (q - 1)/2 = (q + 1)/2$, the weak composition $(j_1, \dots, j_q)$ consisting of $(q + 1)/2$ parts equal to $(b - 1)/2$ and $(q - 1)/2$ parts equal to $(b - 3)/2$ satisfies the required condition.

For each such weak composition, we have

$$\prod_{k=1}^q \binom{b}{2j_k + 1} = \binom{b}{b}^{\frac{q+1}{2}} \binom{b}{b-2}^{\frac{q-1}{2}} = \binom{b}{2}^{\frac{q-1}{2}}.$$

Moreover, the number of such weak compositions is $\binom{q}{(q-1)/2}$. Therefore, restricting (3.2) to these weak compositions, we obtain

$$\begin{aligned} |N(b, q, b)| &\geq \frac{(n - q)!}{2^{q(b-2)}b^{q-1}q!} \binom{q}{\frac{q-1}{2}} \binom{b}{2}^{\frac{q-1}{2}} \\ &= \frac{(n - q)!}{2^{q(b-2)}b^{q-1}q!} \cdot \frac{q!}{\left(\frac{q+1}{2}\right)!\left(\frac{q-1}{2}\right)!} \left(\frac{b(b-1)}{2}\right)^{\frac{q-1}{2}} \\ &= \frac{(n - q)!}{2^{q(b-2)}\left(\frac{q+1}{2}\right)!\left(\frac{q-1}{2}\right)!} \left(\frac{b-1}{2b}\right)^{\frac{q-1}{2}}. \end{aligned}$$

□

3.3. Upper Bound for D

The following is standard.

Proposition 3.5. *Let $g \in S_n$ and suppose that g has cycle type*

$$\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n}).$$

Define the centralizer of g in S_n :

$$Z_g := \{c \in S_n \mid cg = gc\}.$$

Then $|Z_g|$ depends only on the cycle type λ , and

$$|Z_g| = 1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!.$$

Proof. See [Sag01, Proposition 1.1.1]. □

Using this proposition, we obtain the following.

Proposition 3.6. *Let $n \in \mathbb{Z}_{>0}$ with $n \geq 2$, $x \in S_n$ be an n -cycle, and for each $1 \leq k \leq n-1$, define*

$$D_k := \{c \in S_n \mid cx^k = x^k c\}, \tag{3.3}$$

the centralizer of x^k in S_n . Then the following hold.

- (1) $D_1 = \{x^k \mid 0 \leq k \leq n-1\}$.
- (2) For any $1 \leq k, l \leq n-1$, if $k \mid l$, then $D_k \subset D_l$.
- (3) For any $1 \leq k \leq n-1$, let $m = \gcd(n, k)$. Then

$$D_k = D_m,$$

$$|D_k| = |D_m| = \left(\frac{n}{m}\right)^m m!.$$

Proof.

(1) Since x has cycle type

$$(1^0, 2^0, \dots, (n-1)^0, n^1),$$

Proposition 3.5 implies that

$$|D_1| = n^1 \cdot 1! = n.$$

On the other hand,

$$x^0 = \text{id}, x^1, x^2, \dots, x^{n-1}$$

are all distinct, and for every $0 \leq k \leq n-1$,

$$x^k x = x x^k.$$

Hence

$$\{x^k \mid 0 \leq k \leq n-1\} \subset D_1,$$

and the left-hand side contains n elements. Therefore, we conclude that

$$D_1 = \{x^k \mid 0 \leq k \leq n-1\}.$$

(2) Suppose that $k \mid l$. Then $l = dk$ for some $d \in \mathbb{Z}_{>0}$. If $c \in D_k$, then $cx^k = x^k c$, and hence

$$cx^l = c(x^k)^d = (x^k)^d c = x^l c.$$

Hence $c \in D_l$. Since c was arbitrary, we obtain $D_k \subset D_l$.

(3) Since $m \mid k$, (2) implies that

$$D_m \subset D_k. \quad (3.4)$$

We can write $n = mn'$ and $k = mk'$ with $n', k' \in \mathbb{Z}_{>0}$ and $\gcd(n', k') = 1$. Hence there exist $r, s \in \mathbb{Z}$ such that

$$rn' + sk' = 1.$$

Then

$$rn + sk = rmn' + smk' = m(rn' + sk') = m.$$

Since $x^n = 1$, we have

$$x^m = x^{rn+sk} = x^{sk} = (x^k)^s.$$

Let $c \in D_k$. Then

$$cx^m = c(x^k)^s = (x^k)^s c = x^m c,$$

hence $c \in D_m$. Since c was arbitrary, we obtain

$$D_k \subset D_m. \quad (3.5)$$

By (3.4) and (3.5), we conclude that $D_k = D_m$.

Since $m \mid n$, the cycle type of x^m is $((n/m)^m)$. Therefore, by Proposition 3.5,

$$|D_k| = |D_m| = \left(\frac{n}{m}\right)^m m!.$$

□

Corollary 3.7. *Let $x, y \in S_n$, and assume x is an n -cycle. Let $\mathcal{D}$ be the dessin corresponding to the pair (x, y) , whose monodromy group is $\langle x, y \rangle$. Then,*

- (1) $\text{Aut } \mathcal{D} = \{x^k \mid 0 \leq k \leq n-1, x^k y = y x^k\}$.
- (2) $\text{Aut } \mathcal{D}$ is trivial if and only if $x^k y \neq y x^k$ for all $1 \leq k \leq n-1$.

Proof.

(1) $\text{Aut } \mathcal{D}$ is defined by (2.5). By Proposition 3.6(1), the elements commuting with x are precisely the powers of x . Hence

$$\text{Aut } \mathcal{D} = \{x^k \mid 0 \leq k \leq n-1, x^k y = y x^k\}.$$

(2) By (1), we have

$$\text{Aut } \mathcal{D} \text{ is trivial} \iff \text{Aut } \mathcal{D} = \{x^0\} \iff x^k y \neq y x^k \text{ for all } 1 \leq k \leq n-1.$$

□

Proposition 3.8. *Let $n \in \mathbb{Z}_{>0}$ with $n \geq 2$, $x \in S_n$ be an n -cycle, and for each $1 \leq k \leq n-1$, define D_k by (3.3). Let D be the set of permutations $y \in S_n$ such that the corresponding dessin $\mathcal{D}$, with monodromy group $\langle x, y \rangle$, has nontrivial automorphism group. Then*

$$D = \bigcup_{\ell: \text{prime}, \ell \mid n} D_{\frac{n}{\ell}},$$

$$|D| \leq \sum_{\ell: \text{prime}, \ell \mid n} \ell^{\frac{n}{\ell}} \left(\frac{n}{\ell}\right)!.$$

Proof. By Corollary 3.7(2), we have

$$D^c = \bigcap_{k=1}^{n-1} D_k^c,$$

hence

$$D = \bigcup_{k=1}^{n-1} D_k.$$

By Proposition 3.6(2)(3), every D_k is contained in D_m for some maximal proper divisor m of n . Therefore, to determine D and estimate its cardinality, it suffices to consider only those k of the form n/ℓ , where ℓ is a prime divisor of n . Hence

$$D = \bigcup_{\ell: \text{ prime}, \ell | n} D_{\frac{n}{\ell}}.$$

Since $\gcd(n, n/\ell) = n/\ell$, Proposition 3.6(3) yields

$$|D_{\frac{n}{\ell}}| = \ell^{\frac{n}{\ell}} \left(\frac{n}{\ell}\right)!.$$

Consequently,

$$|D| \leq \sum_{\ell: \text{ prime}, \ell | n} |D_{\frac{n}{\ell}}| = \sum_{\ell: \text{ prime}, \ell | n} \ell^{\frac{n}{\ell}} \left(\frac{n}{\ell}\right)!.$$

□

Lemma 3.9. *Let $n \in \mathbb{Z}_{>0}$ with $n \geq 2$, $x \in S_n$ be an n -cycle, and for each $1 \leq k \leq n-1$, define D_k by (3.3). Let*

$$\kappa_1 := \frac{2623}{1894} = 1.384\dots, \quad \kappa_2 := \frac{972}{947} = 1.026\dots$$

Then

$$|D| \leq \kappa_1 \cdot 2^{\frac{n}{2}} \left(\frac{n}{2}\right)!,$$

where, for noninteger M , we define $M! := \Gamma(M+1)$.

In particular, if n is odd, then

$$|D| \leq \kappa_2 \cdot 3^{\frac{n}{3}} \left(\frac{n}{3}\right)!.$$

Proof. When $n = 2$, by Proposition 3.8,

$$|D| = |D_1| = 2^{\frac{n}{2}} \left(\frac{n}{2}\right)! \leq \kappa_1 \cdot 2^{\frac{n}{2}} \left(\frac{n}{2}\right)!.$$

In the following, assume $n \geq 3$.

Let

$$F_n(x) := x^{\frac{n}{x}} \Gamma\left(\frac{n}{x} + 1\right) \quad (x \geq 2). \tag{3.6}$$

Then, for every prime divisor ℓ of n , by Proposition 3.6(3),

$$|D_{\frac{n}{\ell}}| = F_n(\ell).$$

Let $G_n(x) = \log F_n(x)$. Then by (3.6),

$$G_n(x) = \frac{n}{x} \log x + \log \Gamma\left(\frac{n}{x} + 1\right),$$

hence

$$G'_n(x) = -\frac{n}{x^2} \left(\log x - 1 + \psi \left(\frac{n}{x} + 1 \right) \right),$$

where

$$\psi(x) := \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is the digamma function.

It is well known that

$$\psi(x+1) > \log x \quad (x > 0).$$

Hence

$$\begin{aligned} G'_n(x) &< -\frac{n}{x^2} \left(\log x - 1 + \log \frac{n}{x} \right) \\ &= -\frac{n}{x^2} (\log n - 1) \leq -\frac{n}{x^2} (\log 3 - 1) < 0. \end{aligned}$$

Therefore, G_n is strictly decreasing in x . Hence so is F_n .

(i) The case where n has precisely one prime divisor.

Let ℓ denote the unique prime divisor of n . Then we can write $n = \ell^m$ with $m \in \mathbb{Z}_{>0}$. By Proposition 3.8, we have

$$\begin{aligned} D &= D_{\frac{n}{\ell}}, \\ |D| &= \ell^{\frac{n}{\ell}} \left(\frac{n}{\ell} \right)! = F_n(\ell). \end{aligned}$$

Since F_n is decreasing and $\ell \geq 2$, we have

$$|D| = F_n(\ell) \leq F_n(2) = 2^{\frac{n}{2}} \left(\frac{n}{2} \right)! \leq \kappa_1 \cdot 2^{\frac{n}{2}} \left(\frac{n}{2} \right)!.$$

In particular, if n is odd, then $\ell \geq 3$, and therefore

$$|D| = F_n(\ell) \leq F_n(3) = 3^{\frac{n}{3}} \left(\frac{n}{3} \right)! \leq \kappa_2 \cdot 3^{\frac{n}{3}} \left(\frac{n}{3} \right)!.$$

(ii) The case where n has at least 2 prime divisors.

Let

$$\ell_1 < \ell_2 < \cdots < \ell_w \quad (w \geq 2)$$

be the distinct prime divisors of n . For $1 \leq i \leq w-1$, write $n = \ell_i \ell_{i+1} s_i$ with $s_i \in \mathbb{Z}_{>0}$. Since $F_n(\ell) = |D_{n/\ell}|$, we have

$$\frac{F_n(\ell_{i+1})}{F_n(\ell_i)} = \frac{|D_{\frac{n}{\ell_{i+1}}}|}{|D_{\frac{n}{\ell_i}}|} = \frac{\ell_{i+1}^{\frac{n}{\ell_{i+1}}} \left(\frac{n}{\ell_{i+1}} \right)!}{\ell_i^{\frac{n}{\ell_i}} \left(\frac{n}{\ell_i} \right)!} = \frac{\ell_{i+1}^{\ell_i s_i} (\ell_i s_i)!}{\ell_i^{\ell_{i+1} s_i} (\ell_{i+1} s_i)!}.$$

Define

$$H(m, d, s) := \frac{(m+d)^{ms} (ms)!}{m^{(m+d)s} ((m+d)s)!} \quad (m, d, s \in \mathbb{Z}_{>0}, m \geq 2),$$

so that

$$\frac{F_n(\ell_{i+1})}{F_n(\ell_i)} = \frac{|D_{\frac{n}{\ell_{i+1}}}|}{|D_{\frac{n}{\ell_i}}|} = H(\ell_i, \ell_{i+1} - \ell_i, s_i).$$

Then we have

$$\begin{aligned}
\frac{H(m, d, s)}{H(m, d, 1)^s} &= \frac{(m+d)^{ms}(ms)!}{m^{(m+d)s}((m+d)s)!} \cdot \left(\frac{m^{(m+d)}(m+d)!}{(m+d)^m m!} \right)^s \\
&= \frac{(ms)!}{((m+d)s)!} \left(\frac{(m+d)!}{m!} \right)^s \\
&= \frac{(m+1)^s(m+2)^s \cdots (m+d)^s}{(ms+1)(ms+2) \cdots (ms+ds)} \\
&= \frac{(m+1)^s}{(ms+1) \cdots (ms+s)} \cdot \frac{(m+2)^s}{(ms+s+1) \cdots (ms+2s)} \\
&\quad \cdots \frac{(m+d)^s}{(ms+(d-1)s+1) \cdots (ms+ds)} \\
&= \prod_{i=1}^d \frac{(m+i)^s}{\prod_{j=1}^s (ms+(i-1)s+j)}.
\end{aligned}$$

For each $1 \leq i \leq d$ and $1 \leq j \leq s$,

$$ms + (i-1)s + j = s(m+i-1) + j \geq (m+i-1) + 1 = m+i.$$

Hence

$$H(m, d, s) \leq H(m, d, 1)^s. \quad (3.7)$$

Since $(1 + d/m)^m$ is strictly increasing as a function of $m > 0$ and converges to e^d as $m \rightarrow \infty$,

$$\left(\frac{m+d}{m} \right)^m = \left(1 + \frac{d}{m} \right)^m < e^d < m^d \quad (m > e).$$

Hence

$$(m+d)^m < m^{m+d} \quad (m > e).$$

Therefore,

$$H(m, d, 1) = \frac{(m+d)^m}{m^{m+d}} \cdot \frac{1}{(m+1) \cdots (m+d)} < 1 \quad (m \geq 3). \quad (3.8)$$

Moreover, when $m = 2$,

$$\begin{aligned}
H(2, 1, 1) &= \frac{3^2 \cdot 2!}{2^3 \cdot 3!} = \frac{3}{8} < 1, \\
\frac{H(2, d+1, 1)}{H(2, d, 1)} &= \frac{(d+3)^2 \cdot 2!}{2^{d+3}(d+3)!} \cdot \frac{2^{d+2}(d+2)!}{(d+2)^2 \cdot 2!} \\
&= \frac{d+3}{2(d+2)^2} < \frac{d+3}{2(d+2)} < \frac{d+3}{d+4} < 1 \quad (d \geq 1).
\end{aligned}$$

Hence $H(2, d, 1) < 1$ for all $d \geq 1$. Combining this with (3.8), we obtain

$$H(m, d, 1) < 1 \quad (m \geq 2, d \geq 1).$$

By (3.7),

$$H(m, d, s) \leq H(m, d, 1) < 1 \quad (m \geq 2, d \geq 1, s \geq 1).$$

Moreover, regarding $H(m, d, 1)$,

$$\frac{H(m, d+1, 1)}{H(m, d, 1)} = \frac{(m+d+1)^m m!}{m^{m+d+1}(m+d+1)!} \cdot \frac{m^{m+d}(m+d)!}{(m+d)^m m!}$$

$$\begin{aligned}
&= \frac{(m+d+1)^{m-1}}{m(m+d)^m} = \frac{1}{m(m+d)} \left(1 + \frac{1}{m+d}\right)^{m-1} \\
&< \frac{1}{m(m+d)} \left(1 + \frac{1}{m+d}\right)^{m+d}.
\end{aligned}$$

Since

$$\left(1 + \frac{1}{m+d}\right)^{m+d} < e,$$

we obtain

$$\frac{H(m, d+1, 1)}{H(m, d, 1)} < \frac{e}{m(m+d)} \leq \frac{e}{2(2+1)} < 1.$$

Therefore, $H(m, d, 1)$ is strictly decreasing in d .

Moreover,

$$\begin{aligned}
\frac{H(m+1, d, 1)}{H(m, d, 1)} &= \frac{(m+d+1)^{m+1}(m+1)!}{(m+1)^{m+d+1}(m+d+1)!} \cdot \frac{m^{m+d}(m+d)!}{(m+d)^m m!} \\
&= \left(1 + \frac{1}{m+d}\right)^m \left(\frac{m}{m+1}\right)^{m+d} \\
&< \left(1 + \frac{1}{m}\right)^m \left(\frac{m}{m+1}\right)^{m+d} = \left(\frac{m}{m+1}\right)^d < 1.
\end{aligned}$$

Therefore, $H(m, d, 1)$ is also strictly decreasing in m .

Since each ℓ_i is a prime number, we have $\ell_{i+1} - \ell_i = 1$ only when $i = 1$, $\ell_1 = 2$, and $\ell_2 = 3$. Otherwise $\ell_{i+1} - \ell_i \geq 2$. Hence,

$$\begin{aligned}
\frac{F_n(\ell_2)}{F_n(\ell_1)} &= H\left(\ell_1, \ell_2 - \ell_1, \frac{n}{\ell_1 \ell_2}\right) \\
&\leq H(2, 1, 1) = \frac{3^2 \cdot 2!}{2^3 \cdot 3!} = \frac{3}{8},
\end{aligned}$$

and for $i \geq 2$,

$$\begin{aligned}
\frac{F_n(\ell_{i+1})}{F_n(\ell_i)} &= H\left(\ell_i, \ell_{i+1} - \ell_i, \frac{n}{\ell_i \ell_{i+1}}\right) \\
&\leq H(3, 2, 1) = \frac{5^3 \cdot 3!}{3^5 \cdot 5!} = \frac{25}{972}.
\end{aligned}$$

Therefore,

$$F_n(\ell_i) \leq \begin{cases} F_n(2) & (i = 1) \\ \frac{3}{8} \left(\frac{25}{972}\right)^{i-2} F_n(2) & (i \geq 2) \end{cases}.$$

Thus,

$$\begin{aligned}
|D| &\leq \sum_{i=1}^w F_n(\ell_i) \leq F_n(2) + \sum_{i=2}^w \frac{3}{8} \left(\frac{25}{972}\right)^{i-2} F_n(2) \\
&\leq F_n(2) + \frac{3}{8} \sum_{i=2}^{\infty} \left(\frac{25}{972}\right)^{i-2} F_n(2) \\
&= \left(1 + \frac{3}{8} \cdot \frac{1}{1 - \frac{25}{972}}\right) F_n(2) = \frac{2623}{1894} F_n(2)
\end{aligned}$$

$$= \kappa_1 \cdot 2^{\frac{n}{2}} \left(\frac{n}{2}\right)!.$$

When n is odd, we have $\ell_1 \geq 3$ and $\ell_{i+1} - \ell_i \geq 2$ for all i . Hence,

$$\frac{F_n(\ell_{i+1})}{F_n(\ell_i)} \leq H(3, 2, 1) = \frac{25}{972}.$$

Therefore, for all $1 \leq i \leq w$,

$$F_n(\ell_i) \leq \left(\frac{25}{972}\right)^{i-1} F_n(3).$$

Thus,

$$\begin{aligned} |D| &\leq \sum_{i=1}^w F_n(\ell_i) \leq \sum_{i=1}^w \left(\frac{25}{972}\right)^{i-1} F_n(3) \\ &\leq \sum_{i=1}^{\infty} \left(\frac{25}{972}\right)^{i-1} F_n(3) \\ &= \frac{1}{1 - \frac{25}{972}} F_n(3) = \frac{972}{947} F_n(3) \\ &= \kappa_2 \cdot 3^{\frac{n}{3}} \left(\frac{n}{3}\right)!. \end{aligned}$$

Combining (i) and (ii), we obtain

$$|D| \leq \kappa_1 \cdot 2^{\frac{n}{2}} \left(\frac{n}{2}\right)!,$$

and, if n is odd,

$$|D| \leq \kappa_2 \cdot 3^{\frac{n}{3}} \left(\frac{n}{3}\right)!.$$

□

4. Passport $[b^q, b^q, n]$ with Genus ≥ 2

In this section, we consider the passports of the form $[b^q, b^q, n]$, where $n = qb$, $q \geq 2$, and the genus is at least 2. A dessin with such a passport has a single face, equal numbers of black and white vertices, and all vertices have the same valency.

By (2.4), the genus of such a dessin is

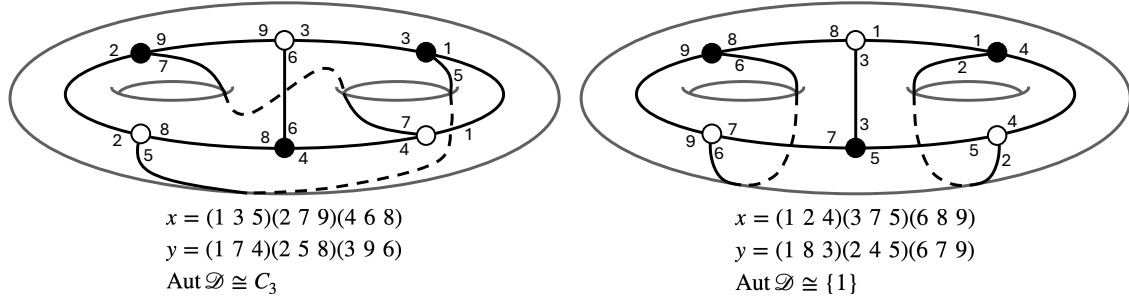
$$g = \frac{n - (2q + 1)}{2} + 1 = \frac{q(b - 2) + 1}{2} \geq 2. \quad (4.1)$$

Since g is an integer, both b and q must be odd. Consequently, n is also odd, and in particular $b, q \geq 3$.

Figure 7 shows examples of dessins d'enfants with passport $[3^3, 3^3, 9]$. The dessin on the left has automorphism group C_3 , while the dessin on the right has trivial automorphism group.

By [Ohn26a, Theorem 6.1], the passport $[a^p, b^q, n]$ has a regular dessin if and only if $\gcd(p, q) = 1$. Since $\gcd(q, q) = q \geq 2$, the passport $[b^q, b^q, n]$ never admits a regular dessin.

We now prove that every uniform passport $[b^q, b^q, n]$ ($q \geq 2$) of genus at least 2 admits a dessin with trivial automorphism group. The proof is divided into two cases, according as $b = 3$ or $b \geq 5$.

FIGURE 7. Dessins d'enfants with passport $[3^3, 3^3, 9]$

4.1. The Subcase $b = 3$

We construct permutations x and y for each passport $[3^q, 3^q, 3^q]$ of genus at least 2 such that the corresponding dessin has a trivial automorphism group. By (4.1), q is odd and $q \geq 3$.

Since the automorphism group is invariant under permutations of the three partitions, it suffices to consider the passport $[3q, 3^q, 3^q]$, that is, the case where x , y , and $z = (xy)^{-1}$ have cycle types $(3q)$, (3^q) , and (3^q) , respectively. Since the automorphism group is also invariant under relabeling of the edges, we may fix x .

Proposition 4.1. *For each odd integer $q \geq 3$, let $x = (1\ 2\ \dots\ 3q) \in S_{3q}$ and define $y \in S_{3q}$ for each q as follows:*

$$y = \alpha\beta_0\beta_1 \cdots \beta_{\frac{q-5}{2}}\gamma,$$

$$\alpha = (1\ 2\ 4)(3\ 7\ 5),$$

$$\beta_i = \begin{cases} (11v+6\ 11v+9\ 11v+10)(11v+8\ 3q-v\ 11v+13) & (i \equiv 0 \pmod{2}) \\ (11v\ 11v+1\ 11v+4)(11v+3\ 11v+7\ 11v+5) & (i \equiv 1 \pmod{2}) \end{cases},$$

$$\text{where } v = \left\lfloor \frac{i+1}{2} \right\rfloor,$$

$$\gamma = \begin{cases} (11w+11\ 11w+12\ 11w+14) & (q \equiv 1 \pmod{4}) \\ (11w+6\ 11w+8\ 11w+9) & (q \equiv 3 \pmod{4}) \end{cases},$$

$$\text{where } w = \left\lfloor \frac{q-3}{4} \right\rfloor. \tag{4.2}$$

Then the following hold:

- (1) y has cycle type (3^q) .
- (2) $(xy)^{-1}$ has cycle type (3^q) .
- (3) For the dessin d'enfant $\mathcal{D}$ corresponding to x and y , we have $\text{Aut } \mathcal{D} \cong \{1\}$.

Remark 4.2. For $q = 3, 5, 7$, and 9 , the permutations y and $(xy)^{-1}$ are as follows:

- $q = 3$:

$$y = (1\ 2\ 4)(3\ 7\ 5)(6\ 8\ 9),$$

$$(xy)^{-1} = (1\ 8\ 3)(2\ 4\ 5)(6\ 7\ 9)$$

This pair (x, y) corresponds to the dessin on the right in Figure 7 after permuting the roles of $y \mapsto x$, and $(xy)^{-1} \mapsto y$.

- $q = 5$:

$$y = (1\ 2\ 4)(3\ 7\ 5)(6\ 9\ 10)(8\ 15\ 13)(11\ 12\ 14),$$

$$(xy)^{-1} = (1\ 8\ 3)(2\ 4\ 5)(6\ 7\ 10)(9\ 13\ 11)(12\ 14\ 15)$$

- $q = 7$:

$$y = (1\ 2\ 4)(3\ 7\ 5)(6\ 9\ 10)(8\ 21\ 13)(11\ 12\ 15)(14\ 18\ 16)(17\ 19\ 20),$$

$$(xy)^{-1} = (1\ 8\ 3)(2\ 4\ 5)(6\ 7\ 10)(9\ 13\ 11)(12\ 15\ 16)(14\ 21\ 19)(17\ 18\ 20)$$

- $q = 9$:

$$y = (1\ 2\ 4)(3\ 7\ 5)(6\ 9\ 10)(8\ 27\ 13)(11\ 12\ 15)(14\ 18\ 16)(17\ 20\ 21)$$

$$(19\ 26\ 24)(22\ 23\ 25),$$

$$(xy)^{-1} = (1\ 8\ 3)(2\ 4\ 5)(6\ 7\ 10)(9\ 13\ 11)(12\ 15\ 16)(14\ 27\ 19)(17\ 18\ 21)$$

$$(20\ 24\ 22)(23\ 25\ 26)$$

These examples illustrate the general pattern of the construction.

Proof. (1) In (4.2), every cycle in y has length 3, and the number of these cycles is

$$2 + 2 \cdot \left(\frac{q-5}{2} + 1 \right) + 1 = q.$$

Therefore, to prove that y has cycle type (3^q) , it suffices to show that every element of $\{1, \dots, 3q\}$ appears in (4.2).

- (i) Case $q \equiv 1 \pmod{4}$.

We have

$$w = \left\lfloor \frac{q-3}{4} \right\rfloor = \frac{q-5}{4},$$

and hence

$$3q = 12w + 15.$$

Moreover, the index i of β_i ranges over

$$0 \leq i \leq \frac{q-5}{2} = 2w.$$

We now verify that every element $s \in \{1, \dots, 3q\} = \{1, \dots, 12w + 15\}$ appears in (4.2).

- If $s \in \{1, 2, 3, 4, 5, 7\}$, then s appears in α .
- If $s \in \{6, 8, 9, 10\}$, then s appears in β_0 .
- If $11 \leq s \leq 11w + 10$, write $s = 11t + u$ with $0 \leq u \leq 10$. Then $1 \leq t \leq w$, and u ranges from 0 to 10 for each t . In this case, s appears in

$$\begin{cases} \beta_{2t-2} & (u = 2) \\ \beta_{2t-1} & (u \in \{0, 1, 3, 4, 5, 7\}). \\ \beta_{2t} & (u \in \{6, 8, 9, 10\}) \end{cases}$$

- The elements $11w + 11$, $11w + 12$, and $11w + 14$ appear in γ .
- The element $11w + 13$ appears in $\beta_{2w} = \beta_{\frac{q-5}{2}}$.
- If $11w + 15 \leq s \leq 12w + 15$, then s appears in $\beta_{2(12w+15-s)}$ as $3q - (12w + 15 - s)$.

Therefore, every element of $\{1, \dots, 3q\}$ appears in (4.2).

- (ii) Case $q \equiv 3 \pmod{4}$.

We have

$$w = \left\lfloor \frac{q-3}{4} \right\rfloor = \frac{q-3}{4},$$

and hence

$$3q = 12w + 9.$$

Moreover, the index i of β_i ranges over

$$0 \leq i \leq \frac{q-5}{2} = 2w-1 \quad (q \geq 7).$$

We now verify that every element $s \in \{1, \dots, 3q\} = \{1, \dots, 12w+9\}$ appears in (4.2).

- If $s \in \{1, 2, 3, 4, 5, 7\}$, then s appears in α .
- The elements $11w+6$, $11w+8$, and $11w+9$ appear in γ .

If $q \geq 7$, then

- If $s \in \{6, 8, 9, 10\}$, then s appears in β_0 .
- If $11 \leq s \leq 11w+5$, write $s = 11t + u$ with $0 \leq u \leq 10$. Then $1 \leq t \leq w$, and u ranges from 0 to 10 when $1 \leq t \leq w-1$ and 0 to 5 when $t = w$. In this case, s appears in

$$\begin{cases} \beta_{2t-2} & (u = 2) \\ \beta_{2t-1} & (u \in \{0, 1, 3, 4, 5, 7\}) \\ \beta_{2t} & (u \in \{6, 8, 9, 10\}) \end{cases}.$$

- The element $11w+7$ appears in $\beta_{2w-1} = \beta_{\frac{q-5}{2}}$.
- If $11w+10 \leq s \leq 12w+9$, then s appears in $\beta_{2(12w+9-s)}$ as $3q - (12w+9-s)$.

Therefore, every element of $\{1, \dots, 3q\}$ appears in (4.2).

By (i) and (ii), we conclude that y has cycle type (3^q) .

(2) Since xy and $(xy)^{-1}$ have the same cycle type, it suffices to show that xy has cycle type (3^q) . We now prove this.

Table 3 lists the q cycles of length 3 in xy arising from (4.2).

We now verify that every element $s \in \{1, \dots, 3q\}$ appears in one of the cycles of xy . Note that

$$3q = \begin{cases} 12w+15 & (q \equiv 1 \pmod{4}) \\ 12w+9 & (q \equiv 3 \pmod{4}) \end{cases}.$$

- If $1 \leq s \leq 5$ or $s = 8$, then s appears in the first two cycles.
- If $6 \leq s \leq 11w+5$ and $s \neq 8$, write $s = 11t + u$ with $0 \leq u \leq 10$. Then, if $u = 0$, we have $t \geq 1$ and s appears as $11(v+1)$ with $v = t-1$. If $u = 2$, we have $t \geq 1$ and s appears as $11v+13$ with $v = t-1$. Otherwise it appears as $11v+u$ with $v = t$.
- If $w \geq 1$, the element $s = 11w+8$ appears as $11v+8$ with $v = w$. If $w = 0$, this element is 8, which has already been accounted for.
- Suppose that $q \equiv 1 \pmod{4}$:
 - If $11w+6 \leq s \leq 11w+14$ and $s \neq 11w+8$, then s appears in the last three cycles.
 - If $s = 11w+15$, then s appears as $3q-w$.
 - If $11w+16 \leq s \leq 12w+15$, then s appears as $3q-(v-1)$, where $v = 12w+16-s$ and $1 \leq v \leq w$.
- Suppose that $q \equiv 3 \pmod{4}$:
 - If $11w+6 \leq s \leq 11w+9$ and $s \neq 11w+8$, then s appears in the last cycle.
 - If $11w+10 \leq s \leq 12w+9$, then s appears as $3q-(v-1)$, where $v = 12w+10-s$ and $1 \leq v \leq w$.

Therefore, every element appears in exactly one cycle, and hence xy has cycle type (3^q) .

s	$y \cdot s$	$xy \cdot s$
1	2	3
3	7	8
8	$3q$	1
2	4	5
5	3	4
4	1	2
$11v + 6$ ($0 \leq v \leq w - 1$)	$11v + 9$	$11v + 10$
$11v + 10$	$11v + 6$	$11v + 7$
$11v + 7$	$11v + 5$	$11v + 6$
$11v + 9$ ($0 \leq v \leq w - 1$)	$11v + 10$	$11v + 11$
$11(v + 1)$	$11(v + 1) + 1$	$11v + 13$
$11v + 13$	$11v + 8$	$11v + 9$
$11v + 1$ ($1 \leq v \leq w$)	$11v + 4$	$11v + 5$
$11v + 5$	$11v + 3$	$11v + 4$
$11v + 4$	$11v$	$11v + 1$
$11v + 3$ ($1 \leq v \leq w$)	$11v + 7$	$11v + 8$
$11v + 8$	$3q - v$	$3q - (v - 1)$
$3q - (v - 1)$	$11(v - 1) + 13$	$11v + 3$
Case $q \equiv 1 \pmod{4}$		
$11w + 6$	$11w + 9$	$11w + 10$
$11w + 10$	$11w + 6$	$11w + 7$
$11w + 7$	$11w + 5$	$11w + 6$
$11w + 9$	$11w + 10$	$11w + 11$
$11w + 11$	$11w + 12$	$11w + 13$
$11w + 13$	$11w + 8$	$11w + 9$
$11w + 12$	$11w + 14$	$3q - w$ ($= 11w + 15$)
$3q - w$	$11w + 13$	$11w + 14$
$11w + 14$	$11w + 11$	$11w + 12$
Case $q \equiv 3 \pmod{4}$		
$11w + 6$	$11w + 8$	$11w + 9$
$11w + 9$	$11w + 6$	$11w + 7$
$11w + 7$	$11w + 5$	$11w + 6$

TABLE 3. Cycles of xy ($s = 1, \dots, 3q$)

(3) By Corollary 3.7(2), it suffices to show that

$$\text{for every } 1 \leq k \leq 3q - 1, \ x^k y \neq yx^k.$$

Equivalently,

$$\text{for every } 1 \leq k \leq 3q - 1,$$

$$\text{there exists } e \in E = \{1, \dots, 3q\} \text{ such that } x^k y \cdot e \neq yx^k \cdot e. \quad (4.3)$$

In what follows, the elements of E are taken modulo $3q$.

(i) Let $e = 3$. Then

$$x^k y \cdot e = x^k \cdot 7 = k + 7,$$

$$yx^k \cdot e = y \cdot (k + 3).$$

The equality $x^k y \cdot e = yx^k \cdot e$ holds only if y maps $k + 3$ to $k + 7$, thus increasing by 4.

From (4.2), the possible values of $k + 3$ are

$$\begin{aligned} 11v + 8, \quad 3q - v \quad (i \equiv 0 \pmod{2}), \\ 11v + 3 \quad (i \equiv 1 \pmod{2}). \end{aligned}$$

We examine these cases separately.

- If $k + 3 = 11v + 8$ and $i \equiv 0 \pmod{2}$, then

$$y \cdot (11v + 8) = 3q - v \equiv 11v + 8 + 4 \pmod{3q},$$

hence

$$12(v + 1) \equiv 0 \pmod{3q},$$

therefore

$$4(v + 1) \equiv 0 \pmod{q}.$$

However, since $i \equiv 0 \pmod{2}$,

$$0 < 4(v + 1) = 4\left(\frac{i}{2} + 1\right) = 2i + 4 \leq 2 \cdot \frac{q - 5}{2} + 4 = q - 1,$$

a contradiction.

- If $k + 3 = 3q - v$ and $i \equiv 0 \pmod{2}$, then

$$y \cdot (3q - v) = 11v + 13 \equiv 3q - v + 4 \pmod{3q},$$

hence

$$12v + 9 \equiv 0 \pmod{3q},$$

therefore

$$4v + 3 \equiv 0 \pmod{q}.$$

However, we have

$$0 < 4v + 3 = 4 \cdot \frac{i}{2} + 3 = 2i + 3 \leq 2 \cdot \frac{q - 5}{2} + 3 = q - 2,$$

a contradiction.

- If $k + 3 = 11v + 3$ and $i \equiv 1 \pmod{2}$, then

$$k = 11v = 11 \left\lfloor \frac{i + 1}{2} \right\rfloor = \frac{11(i + 1)}{2} \quad (1 \leq i \leq (q - 5)/2, \ i: \text{ odd}).$$

Thus, the equality $x^k y \cdot e = y x^k \cdot e$ can hold only if

$$k = 11v = \frac{11(i + 1)}{2} \quad (1 \leq i \leq (q - 5)/2, \ i: \text{ odd}, \ v = (i + 1)/2). \quad (4.4)$$

(ii) Let $e = 1$. By (4.4), it remains to consider $k = 11v = 11(i + 1)/2$ for odd i . (4.2) gives

$$y x^k \cdot e = y \cdot (k + 1) = y \cdot (11v + 1) = 11v + 4.$$

On the other hand,

$$x^k y \cdot e = x^k \cdot 2 = k + 2 = 11v + 2.$$

Hence $x^k y \cdot e \neq y x^k \cdot e$. Therefore, the equality $x^k y \cdot e = y x^k \cdot e$ cannot hold for both $e = 3$ and $e = 1$.

Thus, (4.3) holds. Consequently, for the corresponding dessin d'enfant $\mathcal{D}$, we have $\text{Aut } \mathcal{D} \cong \{1\}$. $\square$

Proposition 4.3. *If a uniform passport $[3^q, 3^q, n]$ ($n = 3q$) has genus at least 2, then it admits a dessin d'enfant with a trivial automorphism group.*

Proof. By (4.1), we have

$$\frac{q+1}{2} \geq 2, \quad q+1 \equiv 0 \pmod{2}.$$

Hence $q \geq 3$ and q is odd.

For such q , let us relabel the permutations y and $(xy)^{-1}$ in Proposition 4.1 as x and y , respectively. Then x , y , and $(xy)^{-1}$ have cycle types (3^q) , (3^q) , and (n) , respectively.

Let $G = \langle x, y \rangle$ be the monodromy group. Then the corresponding dessin $\mathcal{D}$ satisfies $\text{Aut } \mathcal{D} \cong \{1\}$. $\square$

4.2. The Subcase $b \geq 5$

The following lemma will be used in the proof of the case $b \geq 5$.

Lemma 4.4. *Let Γ be the gamma function. If $a > 0$ and $t \geq 1$, then*

$$\Gamma(a+t+1) \leq (a+t)^t \Gamma(a+1).$$

Equality holds if and only if $t = 1$.

Proof. Define

$$\begin{aligned} F(t) &:= \log \frac{(a+t)^t \Gamma(a+1)}{\Gamma(a+t+1)} \\ &= t \log(a+t) + \log \Gamma(a+1) - \log \Gamma(a+t+1). \end{aligned}$$

Then

$$F(1) = \log \frac{(a+1)\Gamma(a+1)}{\Gamma(a+2)} = \log \frac{\Gamma(a+2)}{\Gamma(a+2)} = \log 1 = 0,$$

and

$$F'(t) = \log(a+t) + \frac{t}{a+t} - \psi(a+t+1), \tag{4.5}$$

where

$$\psi(x) := \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is the digamma function.

It is well known that

$$\psi(x+1) < \log x + \frac{1}{x} \quad (x > 0).$$

Hence

$$\psi(a+t+1) < \log(a+t) + \frac{1}{a+t}.$$

Therefore, by (4.5),

$$F'(t) > \log(a+t) + \frac{t}{a+t} - \log(a+t) - \frac{1}{a+t} = \frac{t-1}{a+t}.$$

Since $t \geq 1$, we have $F'(t) \geq 0$, and moreover $F'(t) > 0$ whenever $t > 1$. Therefore, F is increasing on $[1, \infty)$ and strictly increasing on $(1, \infty)$. Hence

$$F(t) \geq F(1) = 0.$$

Therefore,

$$\log \frac{(a+t)^t \Gamma(a+1)}{\Gamma(a+t+1)} \geq 0,$$

and hence

$$\Gamma(a+t+1) \leq (a+t)^t \Gamma(a+1).$$

Equality holds if and only if $t = 1$. □

Proposition 4.5. *Every passport $[b^q, b^q, n]$ of genus ≥ 2 with $b \geq 5$ and $q \geq 2$ admits a dessin with a trivial automorphism group.*

Proof. By the argument in Section 3, it suffices to show that

$$|N(b, q, b)| > |D(n)|,$$

for N and D defined in (3.1).

By Lemma 3.4, we have

$$|N(b, q, b)| \geq \frac{(n-q)!}{2^{q(b-2)} \left(\frac{q+1}{2}\right)! \left(\frac{q-1}{2}\right)!} \left(\frac{b-1}{2b}\right)^{\frac{q-1}{2}},$$

and by Lemma 3.9, since $n = bq$ is odd,

$$|D(n)| \leq \kappa_2 \cdot 3^{\frac{n}{3}} \left(\frac{n}{3}\right)! \quad \left(\left(\frac{n}{3}\right)! := \Gamma\left(\frac{n}{3} + 1\right)\right),$$

where $\kappa_2 = 972/947$. Hence

$$\begin{aligned} \frac{|N(b, q, b)|}{|D(n)|} &\geq \frac{(n-q)!}{2^{q(b-2)} \left(\frac{q+1}{2}\right)! \left(\frac{q-1}{2}\right)!} \left(\frac{b-1}{2b}\right)^{\frac{q-1}{2}} \left(\kappa_2 \cdot 3^{\frac{n}{3}} \left(\frac{n}{3}\right)!\right)^{-1} \\ &= \frac{(q(b-1))!}{\kappa_2 \cdot 2^{q(b-2)} 3^{\frac{bq}{3}} \left(\frac{bq}{3}\right)! \left(\frac{q+1}{2}\right)! \left(\frac{q-1}{2}\right)!} \left(\frac{b-1}{2b}\right)^{\frac{q-1}{2}}. \end{aligned}$$

Let the right-hand side be denoted by $R(b, q)$. We prove that $R(b, q) > 1$ for all $b \geq 5$ and $q \geq 3$. To this end, it suffices to show the following:

- (1) $R(5, 3) > 1$.
- (2) For $b = 5$, $R(b, q)$ is increasing in $q \geq 3$.
- (3) For each fixed $q \geq 3$, $R(b, q)$ is increasing in b .

(1) We have

$$R(5, 3) = \frac{(3(5-1))!}{\kappa_2 \cdot 2^{3(5-2)} 3^{\frac{15}{3}} \left(\frac{15}{3}\right)! \left(\frac{3+1}{2}\right)! \left(\frac{3-1}{2}\right)!} \left(\frac{5-1}{10}\right)^{\frac{3-1}{2}} = \frac{72919}{11664} > 1.$$

(2) We compute

$$\begin{aligned} \frac{R(5, q+2)}{R(5, q)} &= \frac{(4(q+2))!}{2^{3(q+2)} 3^{\frac{5(q+2)}{3}} \left(\frac{5(q+2)}{3}\right)! \left(\frac{q+3}{2}\right)! \left(\frac{q+1}{2}\right)!} \left(\frac{2}{5}\right)^{\frac{q+1}{2}} \\ &\quad \cdot \frac{2^{3q} 3^{\frac{5q}{3}} \left(\frac{5q}{3}\right)! \left(\frac{q+1}{2}\right)! \left(\frac{q-1}{2}\right)!}{(4q)!} \left(\frac{2}{5}\right)^{-\frac{q-1}{2}} \\ &= \frac{\prod_{k=1}^8 (4q+k) \left(\frac{5q}{3}\right)!}{2^6 \cdot 3^{\frac{10}{3}} \left(\frac{5q+10}{3}\right)! \frac{q+1}{2} \cdot \frac{q+3}{2}} \cdot \frac{2}{5} \\ &= \frac{\prod_{k=1}^8 (4q+k)}{40(q+1)(q+3)} \cdot \frac{\left(\frac{5q}{3}\right)!}{3^{\frac{10}{3}} \left(\frac{5q+10}{3}\right)!}. \end{aligned}$$

By Lemma 4.4,

$$\left(\frac{5q+10}{3}\right)! = \Gamma\left(\frac{5q}{3} + \frac{10}{3} + 1\right) \leq \left(\frac{5q+10}{3}\right)^{\frac{10}{3}} \Gamma\left(\frac{5q}{3} + 1\right) = \left(\frac{5q+10}{3}\right)^{\frac{10}{3}} \left(\frac{5q}{3}\right)!.$$

Hence

$$\frac{\left(\frac{5q}{3}\right)!}{3^{\frac{10}{3}} \left(\frac{5q+10}{3}\right)!} \geq \frac{\left(\frac{5q}{3}\right)!}{3^{\frac{10}{3}} \left(\frac{5q+10}{3}\right)^{\frac{10}{3}} \left(\frac{5q}{3}\right)!} = \frac{1}{(5q+10)^{\frac{10}{3}}}.$$

Therefore,

$$\begin{aligned} \frac{R(5, q+2)}{R(5, q)} &\geq \frac{\prod_{k=1}^8 (4q+k)}{40(q+1)(q+3)} \cdot \frac{1}{(5q+10)^{\frac{10}{3}}} \\ &\geq \frac{(4q+1)^8}{40(q+1)(q+3)(5q+10)^{\frac{10}{3}}} \end{aligned}$$

Let the right-hand side be denoted by $f(q)$. Then

$$\begin{aligned} f(3) &= \frac{13^8}{15\sqrt[3]{25} \cdot 10^6} = 18.598 \dots > 1, \\ f'(q) &= \frac{(4q+1)^7(16q^3 + 152q^2 + 388q + 261)}{7500\sqrt[3]{5}(q+1)^2(q+2)^{\frac{13}{3}}(q+3)^2} > 0. \end{aligned}$$

Therefore,

$$\frac{R(5, q+2)}{R(5, q)} > 1 \quad (q \geq 3).$$

Hence $R(5, q)$ is increasing in q .

(3) We have

$$\begin{aligned} \frac{R(b+2, q)}{R(b, q)} &= \frac{(q(b+1))!}{2qb3^{\frac{(b+2)q}{3}} \left(\frac{(b+2)q}{3}\right)! \left(\frac{q+1}{2}\right)! \left(\frac{q-1}{2}\right)!} \left(\frac{b+1}{2(b+2)}\right)^{\frac{q-1}{2}} \\ &\quad \frac{2q(b-2)3^{\frac{bq}{3}} \left(\frac{bq}{3}\right)! \left(\frac{q+1}{2}\right)! \left(\frac{q-1}{2}\right)!}{(q(b-1))!} \left(\frac{b-1}{2b}\right)^{-\frac{q-1}{2}} \\ &= \frac{\prod_{k=1}^{2q} (q(b-1) + k)}{4^q} \cdot \frac{\left(\frac{qb}{3}\right)!}{3^{\frac{2q}{3}} \left(\frac{q(b+2)}{3}\right)!} \left(\frac{b(b+1)}{(b-1)(b+2)}\right)^{\frac{q-1}{2}}. \end{aligned}$$

Since $2q/3 \geq 2 > 1$, by Lemma 4.4 we have

$$\left(\frac{q(b+2)}{3}\right)! = \Gamma\left(\frac{qb}{3} + \frac{2q}{3} + 1\right) \leq \left(\frac{q(b+2)}{3}\right)^{\frac{2q}{3}} \Gamma\left(\frac{qb}{3} + 1\right) = \left(\frac{q(b+2)}{3}\right)^{\frac{2q}{3}} \left(\frac{qb}{3}\right)!.$$

Hence

$$\frac{\left(\frac{qb}{3}\right)!}{3^{\frac{2q}{3}} \left(\frac{q(b+2)}{3}\right)!} \geq \frac{\left(\frac{qb}{3}\right)!}{3^{\frac{2q}{3}} \left(\frac{q(b+2)}{3}\right)^{\frac{2q}{3}} \left(\frac{qb}{3}\right)!} = \frac{1}{(q(b+2))^{\frac{2q}{3}}}.$$

Therefore,

$$\frac{R(b+2, q)}{R(b, q)} \geq \frac{\prod_{k=1}^{2q} (q(b-1) + k)}{4^q} \cdot \frac{1}{q(b+2)^{\frac{2q}{3}}} \left(\frac{b(b+1)}{(b-1)(b+2)}\right)^{\frac{q-1}{2}}.$$

Since

$$\prod_{k=1}^{2q} (q(b-1) + k) \geq (q(b-1) + 1)^{2q}$$

and

$$b(b+1) - (b-1)(b+2) = 2 > 0,$$

we obtain

$$\frac{R(b+2, q)}{R(b, q)} \geq \frac{(q(b-1) + 1)^{2q}}{4^q (q(b+2))^{\frac{2q}{3}}} \cdot 1^{\frac{q-1}{2}} = \left(\frac{(q(b-1) + 1)^3}{8q(b+2)} \right)^{\frac{2q}{3}}.$$

Both the numerator and denominator of the right-hand side are positive. Define

$$g(b, q) := (q(b-1) + 1)^3 - 8q(b+2).$$

Then

$$\begin{aligned} g(5, 3) &= 2029 > 0, \\ \frac{\partial g}{\partial b} &= q(3(q(b-1) + 1)^2 - 8) \geq q(3(3(5-1) + 1)^2 - 8) = 499q > 0, \\ \frac{\partial g}{\partial q} &= 3(b-1)(q(b-1) + 1)^2 - 8(b+2) \geq 3(b-1)(3(b-1) + 1)^2 - 8(b+2) \\ &= 27b^3 - 63b^2 + 40b - 28 = 27(b-5)^3 + 342 \left(b - \frac{1985}{684} \right)^2 + \frac{638471}{1368} > 0. \end{aligned}$$

Hence $g(b, q) > 0$, and therefore

$$\frac{R(b+2, q)}{R(b, q)} > 1 \quad (b \geq 5, q \geq 3).$$

Thus $R(b, q)$ is increasing in b .

By (1), (2), and (3), we conclude that $R(b, q) > 1$ for all $b \geq 5$ and $q \geq 3$. Therefore,

$$|N(b, q, b)| > |D(n)|.$$

This completes the proof. □

Remark 4.6. For $b = 3$, the above argument does not apply.

Indeed, when $b = 3$, we have

$$R(3, 3) = \frac{(3(3-1))!}{\kappa_2 \cdot 2^{3(3-2)} 3^{\frac{9}{3}} \left(\frac{9}{3}\right)! \left(\frac{3+1}{2}\right)! \left(\frac{3-1}{2}\right)!} \left(\frac{3-1}{6}\right)^{\frac{3-1}{2}} = \frac{4735}{52488} < 1,$$

and

$$\begin{aligned} \frac{R(3, q+2)}{R(3, q)} &= \frac{(2(q+2))!}{2^{q+2} 3^{q+2} (q+2)! \left(\frac{q+3}{2}\right)! \left(\frac{q+1}{2}\right)!} \left(\frac{1}{3}\right)^{\frac{q+1}{2}} \cdot \frac{2^q 3^q q! \left(\frac{q+1}{2}\right)! \left(\frac{q-1}{2}\right)!}{(2q)!} \left(\frac{1}{3}\right)^{-\frac{q-1}{2}} \\ &= \frac{4(2q+1)(2q+3)}{27(q+1)(q+3)} = \frac{16}{27} \cdot \frac{q + \frac{1}{2}}{q+1} \cdot \frac{q + \frac{3}{2}}{q+3} < 1 \quad (q \geq 3). \end{aligned}$$

Hence $R(3, q)$ is decreasing in q , and therefore

$$R(3, q) < 1 \quad (q \geq 3).$$

Thus a different approach is required for the case $b = 3$.

4.3. Main Theorem

Combining the results of the previous two subsections, we obtain the following.

Theorem 4.7. *If a uniform passport $[b^q, b^q, n]$ has genus at least 2, then it admits a dessin d'enfant with a trivial automorphism group.*

The same statement holds for the passports $[b^q, n, b^q]$ and $[n, b^q, b^q]$.

Proof. If $q = 1$, then $b = n$, and the statement follows from [Ohn26a, Proposition 7.1].

Assume $q \geq 2$. Since the automorphism group is invariant under permutations of the three partitions, it suffices to consider the passport $[b^q, b^q, n]$.

By (4.1), b is odd and $b \geq 3$. The case $b = 3$ is covered by Proposition 4.3, and the case $b \geq 5$ by Proposition 4.5. $\square$

5. Counterexamples to Previous Conjectures

In [Ohn26a], we conjectured the relationship between the genus of uniform passports and their automorphism groups, as summarized in Table 1. However, we found counterexamples to some of these conjectures. In this section, we present two counterexamples concerning passports of genus at least 2.

5.1. No Dessins with Trivial Automorphism Group

In Table 1, we conjectured that every uniform passport of genus at least 2 admits a dessin d'enfant with trivial automorphism group. However, by direct computation, we found the following counterexample.

- Passport: $[8^2, 2^8, 4^4]$
- Genus: 2
- With x of cycle type (8^2) fixed, there are 920 corresponding permutations y , and every corresponding $\text{Aut } \mathcal{D}$ is nontrivial.

Up to conjugation by $C_{S_n}(x)$, these form 19 conjugacy classes.

If $t \in C_{S_n}(x)$, then $(x, tyt^{-1}) = (txt^{-1}, tyt^{-1})$ defines the same dessin as (x, y) . Therefore, it suffices to count conjugacy classes of such permutations y under conjugation by $C_{S_n}(x)$.

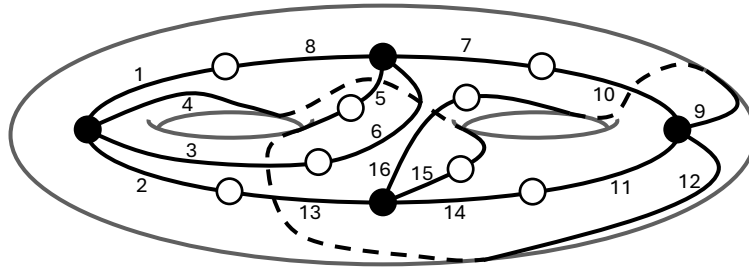
Table 4 lists representatives of the conjugacy classes of y with $x = (1 \dots 8)(9 \dots 16)$ fixed, together with the corresponding orders of automorphism groups and class sizes. There are 11, 6, 1, and 1 conjugacy classes with $|\text{Aut } \mathcal{D}| = 2, 4, 8$, and 16, respectively.

Figure 8 shows a regular dessin with passport $[4^4, 2^8, 8^2]$, obtained from $[8^2, 2^8, 4^4]$ by interchanging the first and third partitions.

This counterexample initially suggested the conjecture that every dessin with passport $[(4t)^2, 2^{4t}, (2t)^4]$ ($t \geq 2$) has a nontrivial automorphism group. However, this conjecture fails already for $t = 3$, since the passport $[12^2, 2^{12}, 6^4]$ admits a dessin with trivial automorphism group. Determining the precise behavior of this family remains an open problem.

$$x = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)(9\ 10\ 11\ 12\ 13\ 14\ 15\ 16)$$

Representative	$ \text{Aut } \mathcal{D} $	Class size
(1 3)(2 5)(4 6)(7 9)(8 10)(11 13)(12 15)(14 16)	2	64
(1 3)(2 9)(4 6)(5 12)(7 14)(8 15)(10 16)(11 13)	2	64
(1 3)(2 9)(4 6)(5 14)(7 11)(8 12)(10 16)(13 15)	2	64
(1 3)(2 9)(4 15)(5 12)(6 13)(7 14)(8 11)(10 16)	2	64
(1 4)(2 5)(3 9)(6 16)(7 13)(8 10)(11 14)(12 15)	2	64
(1 4)(2 6)(3 9)(5 14)(7 16)(8 12)(10 13)(11 15)	2	64
(1 4)(2 7)(3 9)(5 11)(6 14)(8 16)(10 13)(12 15)	2	64
(1 4)(2 9)(3 7)(5 14)(6 10)(8 12)(11 15)(13 16)	2	64
(1 4)(2 9)(3 14)(5 12)(6 15)(7 16)(8 11)(10 13)	2	64
(1 9)(2 10)(3 12)(4 14)(5 15)(6 11)(7 16)(8 13)	2	64
(1 9)(2 10)(3 13)(4 15)(5 11)(6 16)(7 12)(8 14)	2	64
(1 3)(2 6)(4 9)(5 7)(8 13)(10 12)(11 15)(14 16)	4	32
(1 3)(2 9)(4 11)(5 7)(6 13)(8 15)(10 16)(12 14)	4	32
(1 3)(2 9)(4 15)(5 7)(6 13)(8 11)(10 16)(12 14)	4	32
(1 5)(2 6)(3 9)(4 14)(7 13)(8 10)(11 15)(12 16)	4	32
(1 5)(2 9)(3 12)(4 15)(6 13)(7 16)(8 11)(10 14)	4	32
(1 9)(2 10)(3 11)(4 16)(5 13)(6 14)(7 15)(8 12)	4	32
(1 5)(2 9)(3 7)(4 15)(6 13)(8 11)(10 14)(12 16)	8	16
(1 9)(2 12)(3 15)(4 10)(5 13)(6 16)(7 11)(8 14)	16	8

TABLE 4. Representatives of conjugacy classes of y for the passport $[8^2, 2^8, 4^4]$ 

$$x = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8)(9\ 10\ 11\ 12)(13\ 14\ 15\ 16)$$

$$y = (1\ 8)(2\ 13)(3\ 6)(4\ 15)(5\ 12)(7\ 10)(9\ 16)(11\ 14)$$

$$\text{Aut } \mathcal{D} \cong SD_{16}$$

FIGURE 8. A regular dessin d'enfant with passport $[4^4, 2^8, 8^2]$

5.2. Only Dessins with Trivial Automorphism Group

In Table 1, we conjectured that every uniform passport of genus at least 2 with at most one (n) -cycle admits a nonregular dessin d'enfant with nontrivial automorphism group, that is, a dessin $\mathcal{D}$ with $1 < |\text{Aut } \mathcal{D}| < n$. However, a direct computation revealed the following counterexample:

- Passport: $[15, 5^3, 5^3]$
- Genus: 5
- For a fixed 15-cycle x , we have $|N(5, 3, 5)| = 4354560$, and every corresponding dessin has trivial automorphism group.
- The number of conjugacy classes of such y is 290304, and each conjugacy class has 15 elements.

Figure 9 shows one of the dessins with passport $[5^3, 5^3, 15]$, obtained from $[15, 5^3, 5^3]$ by interchanging the first and third partitions.

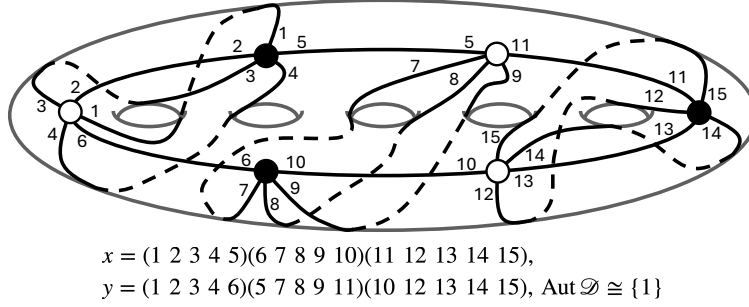


FIGURE 9. A dessin d'enfant with passport $[5^3, 5^3, 15]$

The above passports are special cases of the following theorem.

Theorem 5.1. *Let ℓ_1 and ℓ_2 be distinct prime numbers with $\ell_1 < \ell_2$. Then every dessin d'enfant with passport $[\ell_2^{\ell_1}, \ell_2^{\ell_1}, \ell_1 \ell_2]$ has trivial automorphism group.*

The same conclusion holds for the passports $[\ell_1 \ell_2, \ell_2^{\ell_1}, \ell_2^{\ell_1}]$ and $[\ell_2^{\ell_1}, \ell_1 \ell_2, \ell_2^{\ell_1}]$.

Proof. Let $n = \ell_1 \ell_2$. Since the automorphism group is invariant under permutations of the three partitions, it suffices to consider the passport $[\ell_1 \ell_2, \ell_2^{\ell_1}, \ell_2^{\ell_1}]$. Moreover, since the automorphism group is also invariant under relabeling of the edges, we may assume that $x = (1\ 2\ \dots\ n)$.

By (2.4), the genus is

$$\frac{n - (2\ell_1 + 1)}{2} + 1 = \frac{\ell_1(\ell_2 - 2) + 1}{2}.$$

Since the genus is an integer, both ℓ_1 and ℓ_2 must be odd. Hence $\ell_1 \geq 3$, $\ell_2 \geq 5$, and the genus is at least 5.

Let $y \in S_n$ be a permutation of cycle type $(\ell_2^{\ell_1})$ corresponding to the second partition of the passport. Then $(xy)^{-1}$ also has cycle type $(\ell_2^{\ell_1})$. Since a permutation and its inverse have the same cycle type, xy also has cycle type $(\ell_2^{\ell_1})$.

Assume, for contradiction, that the dessin corresponding to (x, y) has nontrivial automorphism group.

As in Proposition 3.8, let D be the set of permutations in S_n such that the corresponding dessin has nontrivial automorphism group. Then we have $y \in D$.

By Proposition 3.8,

$$D = D_{\frac{n}{\ell_2}} \cup D_{\frac{n}{\ell_1}} = D_{\ell_1} \cup D_{\ell_2}.$$

Hence y belongs to either D_{ℓ_1} or D_{ℓ_2} .

(i) Assume that $y \in D_{\ell_1}$.

By the definition (3.3), we have

$$yx^{\ell_1} = x^{\ell_1}y. \tag{5.1}$$

For $e \in E = \{1, \dots, n\}$, denote by $O_1(e)$ the orbit of e under x^{ℓ_1} . Then

$$O_1(e) = \{e, e + \ell_1, e + 2\ell_1, \dots, e + (\ell_2 - 1)\ell_1\},$$

where the entries are taken modulo n . Therefore, E is partitioned into the following ℓ_1 distinct orbits:

$$B_1 := O_1(1) = \{1, 1 + \ell_1, \dots, 1 + (\ell_2 - 1)\ell_1\},$$

$$\begin{aligned}
 B_2 &:= O_1(2) = \{2, 2 + \ell_1, \dots, 2 + (\ell_2 - 1)\ell_1\}, \\
 &\dots \\
 B_{\ell_1} &:= O_1(\ell_1) = \{\ell_1, 2\ell_1, \dots, \ell_2\ell_1\},
 \end{aligned}$$

By (5.1), for $e \in E$ and $m \in \mathbb{Z}$ we have

$$y \cdot ((x^{\ell_1})^m \cdot e) = (x^{\ell_1})^m \cdot (y \cdot e).$$

Hence

$$y \cdot O_1(e) = O_1(y \cdot e).$$

Therefore, for any $1 \leq i \leq \ell_1$, we have $y \cdot B_i = B_j$ for some $1 \leq j \leq \ell_1$. In other words, y permutes the subsets $B_1, \dots, B_{\ell_1}$. Thus the partition

$$\mathcal{B} := \{B_1, \dots, B_{\ell_1}\}$$

is preserved by y , and hence forms a block system for the action of y on E .

Therefore, y induces a permutation τ on the set of blocks $\mathcal{B}$, and hence $\tau \in S_{\ell_1}$. Since y has cycle type $(\ell_2^{\ell_1})$, we have

$$y^{\ell_2} = \text{id}_{S_n},$$

hence

$$\tau^{\ell_2} = \text{id}_{S_{\ell_1}}.$$

Therefore, $\text{ord}(\tau) \mid \ell_2$. Since ℓ_2 is prime,

$$\text{ord}(\tau) = 1 \text{ or } \ell_2.$$

On the other hand, since $\tau \in S_{\ell_1}$, we have

$$\text{ord}(\tau) \mid |S_{\ell_1}| = \ell_1!.$$

Since ℓ_2 is a prime greater than ℓ_1 , we have $\ell_2 \nmid \ell_1!$. Therefore $\text{ord}(\tau) \neq \ell_2$, and hence

$$\text{ord}(\tau) = 1.$$

Consequently,

$$y \cdot B_i = B_i \quad (1 \leq i \leq \ell_1). \tag{5.2}$$

Since xy also has cycle type $(\ell_2^{\ell_1})$, we have

$$(xy)^{\ell_2} = \text{id}_{S_n}.$$

Hence, on the block system $\mathcal{B}$,

$$(xy)^{\ell_2} \cdot B_i = B_i \quad (1 \leq i \leq \ell_1). \tag{5.3}$$

On the other hand, by (5.2), y preserves every block B_i . Hence

$$xy \cdot B_i = x \cdot (y \cdot B_i) = x \cdot B_i.$$

Therefore, the induced action of xy on $\mathcal{B}$ coincides with that of x , and thus

$$(xy)^{\ell_2} \cdot B_i = x^{\ell_2} \cdot B_i.$$

Let $B_j = x^{\ell_2} \cdot B_i$. Since x cyclically permutes the blocks,

$$x \cdot B_i = \begin{cases} B_{i+1} & (1 \leq i \leq \ell_1 - 1) \\ B_1 & (i = \ell_1) \end{cases},$$

we obtain

$$j \equiv i + \ell_2 \pmod{\ell_1}. \tag{5.4}$$

By (5.3) and (5.4),

$$i \equiv i + \ell_2 \pmod{\ell_1},$$

and hence $\ell_1 \mid \ell_2$. This contradicts the assumption that ℓ_1 and ℓ_2 are distinct primes. Therefore $y \notin D_{\ell_1}$.

(ii) Assume that $y \in D_{\ell_2}$.

By the definition (3.3), we have

$$yx^{\ell_2} = x^{\ell_2}y. \quad (5.5)$$

For $e \in E = \{1, \dots, n\}$, denote by $O_2(e)$ the orbit of e under x^{ℓ_2} . Then

$$O_2(e) = \{e, e + \ell_2, e + 2\ell_2, \dots, e + (\ell_1 - 1)\ell_2\},$$

where the entries are taken modulo n . Therefore, E is partitioned into the following ℓ_2 distinct orbits:

$$C_1 := O_2(1) = \{1, 1 + \ell_2, \dots, 1 + (\ell_1 - 1)\ell_2\},$$

$$C_2 := O_2(2) = \{2, 2 + \ell_2, \dots, 2 + (\ell_1 - 1)\ell_2\},$$

...

$$C_{\ell_2} := O_2(\ell_2) = \{\ell_2, 2\ell_2, \dots, \ell_1\ell_2\},$$

By (5.5), for $e \in E$ and $m \in \mathbb{Z}$ we have

$$y \cdot ((x^{\ell_2})^m \cdot e) = (x^{\ell_2})^m \cdot (y \cdot e).$$

Hence

$$y \cdot O_2(e) = O_2(y \cdot e).$$

Therefore, for any $1 \leq i \leq \ell_2$, $y \cdot C_i = C_j$ for some $1 \leq j \leq \ell_2$. Thus the set

$$\mathcal{C} := \{C_1, \dots, C_{\ell_2}\}$$

forms a block system preserved by y .

Therefore, y induces a permutation π on the set of blocks $\mathcal{C}$, and hence $\pi \in S_{\ell_2}$.

For $e \in E$, write

$$e = (i, t),$$

where $e \in C_i$ and $e = i + t\ell_2 \pmod{n}$. Then $1 \leq i \leq \ell_2$ and we may regard t as an element of $\mathbb{Z}/\ell_1\mathbb{Z}$. Then

$$x^{\ell_2} \cdot (i, t) = (i, t + 1).$$

Therefore, there exists a map $f_i : \mathbb{Z}/\ell_1\mathbb{Z} \rightarrow \mathbb{Z}/\ell_1\mathbb{Z}$ such that

$$y \cdot (i, t) = (\pi(i), f_i(t)). \quad (5.6)$$

Moreover,

$$yx^{\ell_2} \cdot (i, t) = y \cdot (i, t + 1) = (\pi(i), f_i(t + 1)),$$

$$x^{\ell_2}y \cdot (i, t) = x^{\ell_2}(\pi(i), f_i(t)) = (\pi(i), f_i(t) + 1).$$

By (5.5),

$$(\pi(i), f_i(t + 1)) = (\pi(i), f_i(t) + 1),$$

hence

$$f_i(t + 1) = f_i(t) + 1 \quad (t \in \mathbb{Z}/\ell_1\mathbb{Z}).$$

Let $s_i = f_i(0)$. Then we obtain

$$\begin{aligned} f_i(1) &= f_i(0) + 1 = s_i + 1, \\ f_i(2) &= f_i(1) + 1 = s_i + 2, \\ &\dots \\ f_i(\ell_1 - 1) &= f_i(\ell_1 - 2) + 1 = s_i + (\ell_1 - 1). \end{aligned}$$

Hence

$$f_i(t) = s_i + t \quad (t \in \mathbb{Z}/\ell_1\mathbb{Z}).$$

Substituting this into (5.6), we obtain

$$y \cdot (i, t) = (\pi(i), s_i + t). \quad (5.7)$$

Thus y maps each block C_i to $C_{\pi(i)}$ and acts on the second coordinate by a translation modulo ℓ_1 .

Since y has cycle type $(\ell_2^{\ell_1})$, we have

$$y^{\ell_2} = \text{id}_{S_n}. \quad (5.8)$$

Since π is the permutation induced by y on the block system $\mathcal{C}$, it follows that

$$\pi^{\ell_2} = \text{id}_{S_{\ell_2}}.$$

Hence $\text{ord}(\pi) \mid \ell_2$. Since ℓ_2 is prime, we have

$$\text{ord}(\pi) = 1 \text{ or } \ell_2.$$

If $\text{ord}(\pi) = 1$, then $\pi = \text{id}_{S_{\ell_2}}$ and hence

$$y \cdot (i, t) = (i, s_i + t).$$

Thus y preserves each block C_i . Therefore every cycle of y is contained in a block of size ℓ_1 , and hence has length at most ℓ_1 . This contradicts the fact that every cycle of y has length ℓ_2 . Therefore,

$$\text{ord}(\pi) = \ell_2.$$

By (5.8), we have

$$y^{\ell_2} \cdot (i, t) = (i, t). \quad (5.9)$$

Since $\text{ord}(\pi) = \ell_2$, the orbit of every block under π consists of all blocks in $\mathcal{C}$. Hence, starting from (i, t) and applying y repeatedly, the block component visits each block C_j exactly once before returning to C_i . Therefore each translation parameter s_j is added exactly once, and hence

$$y^{\ell_2} \cdot (i, t) = \left(i, \sum_{j=1}^{\ell_2} s_j + t \right). \quad (5.10)$$

By (5.9) and (5.10),

$$\sum_{j=1}^{\ell_2} s_j \equiv 0 \pmod{\ell_1}. \quad (5.11)$$

The action of x is given by

$$x \cdot (i, t) = \begin{cases} (i+1, t) & (i \neq \ell_2) \\ (1, t+1) & (i = \ell_2) \end{cases}.$$

Hence, together with (5.7), we obtain

$$xy \cdot (i, t) = \begin{cases} (\pi(i) + 1, s_i + t) & (\pi(i) \neq \ell_2) \\ (1, s_i + t + 1) & (\pi(i) = \ell_2) \end{cases}.$$

Since xy has cycle type $(\ell_2^{\ell_1})$, we have

$$(xy)^{\ell_2} = \text{id}_{S_n},$$

and hence

$$(xy)^{\ell_2} \cdot (i, t) = (i, t). \quad (5.12)$$

Define a permutation π' by

$$\pi'(i) = \begin{cases} \pi(i) + 1 & (\pi(i) \neq \ell_2) \\ 1 & (\pi(i) = \ell_2) \end{cases},$$

equivalently,

$$\pi'(i) = \begin{cases} \pi(i) + 1 & (i \neq \pi^{-1}(\ell_2)) \\ 1 & (i = \pi^{-1}(\ell_2)) \end{cases}.$$

Then

$$xy \cdot (i, t) = \begin{cases} (\pi'(i), s_i + t) & (i \neq \pi^{-1}(\ell_2)) \\ (\pi'(i), s_i + t + 1) & (i = \pi^{-1}(\ell_2)). \end{cases}$$

Clearly, $\pi' \in S_{\ell_2}$. Since $(xy)^{\ell_2} = \text{id}_{S_n}$ and π' is the permutation induced by xy on the block system $\mathcal{C}$, we have

$$(\pi')^{\ell_2} = \text{id}_{S_{\ell_2}}.$$

Hence $\text{ord}(\pi') \mid \ell_2$. Since ℓ_2 is prime,

$$\text{ord}(\pi') = 1 \text{ or } \ell_2.$$

As in the case of π , the equality $\text{ord}(\pi') = 1$ leads to a contradiction. Therefore,

$$\text{ord}(\pi') = \ell_2.$$

Thus, the orbit of (i, t) under xy visits all blocks C_i exactly once before returning to the initial block. Therefore, the translations by the s_j are accumulated once each, and in addition, 1 is added once, namely when the orbit passes through the block $C_{\pi^{-1}(\ell_2)}$. Hence

$$(xy)^{\ell_2} \cdot (i, t) = \left(i, \sum_{j=1}^{\ell_2} s_j + 1 + t \right). \quad (5.13)$$

By (5.12) and (5.13),

$$\sum_{j=1}^{\ell_2} s_j + 1 \equiv 0 \pmod{\ell_1}.$$

This contradicts (5.11). Therefore $y \notin D_{\ell_2}$.

By (i) and (ii), neither $y \in D_{\ell_1}$ nor $y \in D_{\ell_2}$ is possible. Since $D = D_{\ell_1} \cup D_{\ell_2}$, we have $y \notin D$. Therefore the corresponding dessin has trivial automorphism group. $\square$

Accordingly, in addition to $[15, 5^3, 5^3]$, each of the passports

$$[21, 7^3, 7^3], [33, 11^3, 11^3], [35, 7^5, 7^5], [39, 13^3, 13^3], [51, 17^3, 17^3], [55, 11^5, 11^5], \dots$$

has the property that every corresponding dessin has trivial automorphism group.

6. Passport $[n, n, n]$ with Genus ≥ 2

In this section, we consider passports of the form $[n, n, n]$ with genus at least 2. A dessin with such a passport has one black vertex, one white vertex, and one face.

By (2.4), the genus is

$$g = \frac{n-3}{2} + 1 = \frac{n-1}{2} \geq 2.$$

Since g is an integer, n is odd and $n \geq 5$.

By [Ohn26a, Corollary 6.2], this passport always admits a regular dessin. On the other hand, by [Ohn26a, Proposition 7.1], it always admits a dessin with trivial automorphism group. We investigate the intermediate case, namely whether the passport admits a nonregular dessin with nontrivial automorphism group.

6.1. Nonregular Dessins with Nontrivial Automorphism Group

Theorem 6.1. *Let $[n, n, n]$ be a uniform passport of genus at least 2. Then it admits a nonregular dessin d'enfant with nontrivial automorphism group if and only if n is composite.*

Furthermore, if n is composite, then for every divisor r of n satisfying $1 < r < n$, there exists a dessin $\mathcal{D}$ with passport $[n, n, n]$ such that $|\text{Aut } \mathcal{D}| = r$.

Proof. Assume that n is prime. Since $|\text{Aut } \mathcal{D}|$ divides the number of edges n , we have $|\text{Aut } \mathcal{D}| = 1$ or n . If $|\text{Aut } \mathcal{D}| = 1$, then the dessin has trivial automorphism group. If $|\text{Aut } \mathcal{D}| = n$, then the dessin is regular, as noted after Proposition 2.9. Hence this passport admits no nonregular dessin with nontrivial automorphism group.

Thus, the existence of a nonregular dessin with nontrivial automorphism group implies that n is composite.

Conversely, assume that n is composite. Let r be a divisor of n satisfying $1 < r < n$. Then we may write $n = rs$. Since n is odd, both r and s are odd integers, and $r, s \geq 3$.

Fix $x = (1\ 2\ \dots\ n)$ and let

$$\begin{aligned} t &= (s\ 2s\ \dots\ rs(=n)), \\ y &= txt^{-1}. \end{aligned}$$

Explicitly, y is obtained from x by cyclically permuting the entries $s, 2s, \dots, (r-1)s, rs(=n)$. Hence y also has cycle type (n) . Let $\mathcal{D}$ denote the dessin corresponding to the pair (x, y) .

We show that

- (1) $(xy)^{-1}$ has cycle type (n) , and
- (2) For the dessin $\mathcal{D}$ corresponding to x and y , we have $|\text{Aut } \mathcal{D}| = r$.

In what follows, all indices are taken modulo n .

- (1) Since xy and $(xy)^{-1}$ have the same cycle type, it suffices to show that xy has cycle type (n) . For each $e \in E = \{1, \dots, n\}$, we can uniquely write $e = is + j$ with $0 \leq i \leq r-1$ and $1 \leq j \leq s$.

If $j \neq s$, then e is fixed by both t and t^{-1} . On the other hand, t sends $is + s$ to $(i+1)s + s$, while t^{-1} sends $is + s$ to $(i-1)s + s$.

Moreover, x sends $is + j$ to $is + j + 1$.

Therefore, for $1 \leq j \leq s-2$, the action of $xy = txt^{-1}$ on e is

$$is + j \xrightarrow{t^{-1}} is + j \xrightarrow{x} is + j + 1 \xrightarrow{t} is + j + 1 \xrightarrow{x} is + j + 2,$$

hence $xy \cdot e = e + 2$.

For $j = s - 1$, we have

$$is + s - 1 \xrightarrow{t^{-1}} is + s - 1 \xrightarrow{x} is + s \xrightarrow{t} (i + 1)s + s \xrightarrow{x} (i + 2)s + 1,$$

hence $xy \cdot e = e + s + 2$.

For $j = s$, we have

$$is + s \xrightarrow{t^{-1}} (i - 1)s + s \xrightarrow{x} is + 1 \xrightarrow{t} is + 1 \xrightarrow{x} is + 2,$$

hence $xy \cdot e = e - s + 2$.

Thus, since s is odd and $s \geq 3$, starting from $is + 1$ and applying xy repeatedly, we obtain

$$\begin{aligned} is + 1 &\xrightarrow{xy} is + 3 \rightarrow \cdots \rightarrow is + s - 2 \rightarrow is + s \\ &\rightarrow is + 2 \rightarrow is + 4 \rightarrow \cdots \rightarrow is + s - 1 \rightarrow (i + 2)s + 1. \end{aligned}$$

Each element $is + j$ ($1 \leq j \leq s$) appears exactly once in this sequence, and the last element is mapped to $(i + 2)s + 1$.

Since $\gcd(r, 2) = 1$, repeated addition of 2 modulo r visits every residue class modulo r . Hence the orbit passes through all elements of E before returning to its starting point. Therefore, xy has cycle type (n) .

(2) By Corollary 3.7(1), we have

$$\text{Aut } \mathcal{D} = \{x^k \mid 0 \leq k \leq n - 1, x^k y = yx^k\}. \quad (6.1)$$

x^k sends $is + j$ to $is + j + k$ modulo n .

Assume that $1 \leq k \leq s - 1$ and let $e = rs = n$. The image of e under $x^k y = x^k txt^{-1}$ is

$$rs \xrightarrow{t^{-1}} (r - 1)s \xrightarrow{x} (r - 1)s + 1 \xrightarrow{t} (r - 1)s + 1 \xrightarrow{x^k} (r - 1)s + k + 1.$$

On the other hand, the image of e under $yx^k = txt^{-1}x^k$ is

$$rs \xrightarrow{x^k} rs + k \xrightarrow{t^{-1}} rs + k \xrightarrow{x} rs + k + 1 \xrightarrow{t} \begin{cases} rs + k + 1 & (k \neq s - 1) \\ 2s & (k = s - 1) \end{cases}.$$

Since

$$(r - 1)s + k + 1 \neq rs + k + 1 \quad (k \neq s - 1)$$

and, when $k = s - 1$,

$$(r - 1)s + k + 1 = (r - 1)s + s = rs \neq 2s,$$

the images of e under $x^k y$ and yx^k differ. Therefore,

$$x^k y \neq yx^k \quad (1 \leq k \leq s - 1). \quad (6.2)$$

We next show that x^s commutes with y . To this end, consider the actions of $x^s y = x^s txt^{-1}$ and $yx^s = txt^{-1}x^s$ on $e = is + j$ ($0 \leq i \leq r - 1$, $1 \leq j \leq s$).

For $1 \leq j \leq s - 2$, we have

$$\begin{aligned} x^s y: is + j &\xrightarrow{t^{-1}} is + j \xrightarrow{x} is + j + 1 \xrightarrow{t} is + j + 1 \\ &\xrightarrow{x^s} (i + 1)s + j + 1, \\ yx^s: is + j &\xrightarrow{x^s} (i + 1)s + j \\ &\xrightarrow{t^{-1}} (i + 1)s + j \xrightarrow{x} (i + 1)s + j + 1 \xrightarrow{t} (i + 1)s + j + 1. \end{aligned}$$

For $j = s - 1$, we have

$$\begin{aligned} x^s y: is + s - 1 &\xrightarrow{t^{-1}} is + s - 1 \xrightarrow{x} is + s \xrightarrow{t} (i + 1)s + s \\ &\xrightarrow{x^s} (i + 2)s + s, \\ yx^s: is + s - 1 &\xrightarrow{x^s} (i + 1)s + s - 1 \\ &\xrightarrow{t^{-1}} (i + 1)s + s - 1 \xrightarrow{x} (i + 1)s + s \xrightarrow{t} (i + 2)s + s. \end{aligned}$$

For $j = s$, we have

$$\begin{aligned} x^s y: is + s &\xrightarrow{t^{-1}} (i - 1)s + s \xrightarrow{x} is + 1 \xrightarrow{t} is + 1 \\ &\xrightarrow{x^s} (i + 1)s + 1, \\ yx^s: is + s &\xrightarrow{x^s} (i + 1)s + s \\ &\xrightarrow{t^{-1}} is + s \xrightarrow{x} (i + 1)s + 1 \xrightarrow{t} (i + 1)s + 1. \end{aligned}$$

Thus, $x^s y$ and yx^s agree on every element of E . Therefore,

$$x^s y = yx^s. \quad (6.3)$$

For $0 \leq k \leq n - 1 = rs - 1$, write $k = us + v$ ($0 \leq u \leq r - 1$, $0 \leq v \leq s - 1$). When $v = 0$, by (6.3),

$$x^k y = (x^s)^u y = y(x^s)^u = yx^k.$$

On the other hand, when $1 \leq v \leq s - 1$, by (6.2) we have $x^v y \neq yx^v$. Since right multiplication by $(x^s)^u$ is bijective, we obtain

$$x^k y = x^v (x^s)^u y = x^v y (x^s)^u \neq yx^v (x^s)^u = yx^k.$$

Hence we obtain

$$x^k y = yx^k \iff s \mid k.$$

Then by (6.1),

$$|\text{Aut } \mathcal{D}| = \frac{n}{s} = r.$$

Since $1 < |\text{Aut } \mathcal{D}| = r < n$, the dessin $\mathcal{D}$ is nonregular and has a nontrivial automorphism group.

Since r was arbitrary, the second assertion of the theorem also follows. $\square$

Figure 10 shows a dessin d'enfant with passport $[9, 9, 9]$ of genus 4 and $|\text{Aut } \mathcal{D}| = 3$.

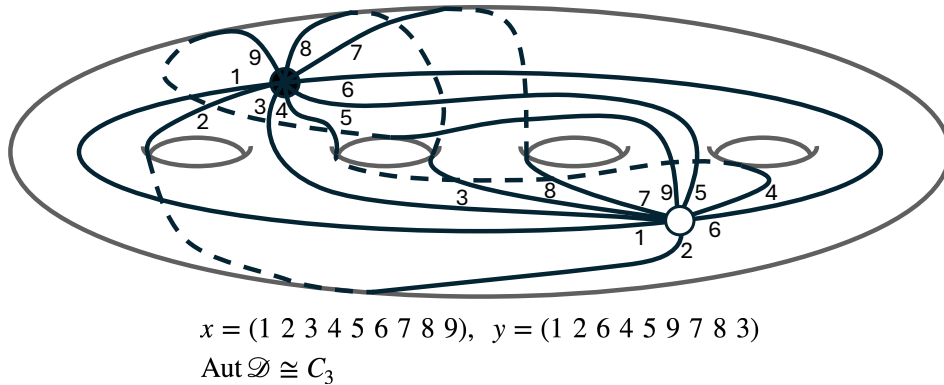


FIGURE 10. A dessin d'enfant with passport $[9, 9, 9]$ and $|\text{Aut } \mathcal{D}| = 3$

Combining the above theorem with the cases of regular dessins and dessins with trivial automorphism group, we obtain the following corollary.

Corollary 6.2. *Let $[n, n, n]$ be a uniform passport of genus at least 2. Then it admits a dessin d'enfant $\mathcal{D}$ with $|\text{Aut } \mathcal{D}| = r$ if and only if r is a divisor of n .*

7. Alternative Proof of the $[n, b^q, n]$ Case

In [Ohn26a, Theorem 7.6], we proved that every passport $[n, b^q, n]$ with $n = bq$ and genus at least 2 admits a dessin with trivial automorphism group. In this section, we present an alternative proof using counting arguments based on centralizers of powers of an n -cycle.

For the passport $[n, n, n]$, that is, the case $q = 1$, [Ohn26a, Proposition 7.1] gives an explicit construction of x and y with $\text{Aut } \mathcal{D} \cong \{1\}$.

The following two subsections treat separately the cases $b = 2$ (where necessarily $q \geq 4$) and $b \geq 3$ with $q \geq 2$.

7.1. The Subcase $b = 2$

We proved the $b = 2$ case in [Ohn26a, Proposition 7.3]. Here we give another proof by explicitly constructing a pair (x, y) for each q .

Proposition 7.1. *For each even integer $q \geq 4$, fix $x = (1\ 2\ \dots\ 2q) \in S_{2q}$ and define $y \in S_{2q}$ for each q as follows:*

$$\begin{aligned} y &= \alpha_1 \alpha_2 \cdots \alpha_{q/2-2} \beta, \\ \alpha_i &= (4i-3\ 4i-1)(4i-2\ 4i) \quad (1 \leq i \leq q/2-2), \\ \beta &= (2q-7\ 2q-5)(2q-6\ 2q-3)(2q-4\ 2q-1)(2q-2\ 2q). \end{aligned} \tag{7.1}$$

Then the following hold:

- (1) y has cycle type (2^q) .
- (2) $(xy)^{-1}$ has cycle type $(2q)$.
- (3) For the dessin d'enfant $\mathcal{D}$ corresponding to x and y , we have $\text{Aut } \mathcal{D} \cong \{1\}$.

Remark 7.2. For $q = 4, 6, 8$, and 10 , the permutations y and $(xy)^{-1}$ are as follows:

- $q = 4$:

$$\begin{aligned} y &= (1\ 3)(2\ 5)(4\ 7)(6\ 8), \\ (xy)^{-1} &= (1\ 6\ 2\ 3\ 5\ 7\ 8\ 4) \end{aligned}$$

- $q = 6$:

$$\begin{aligned} y &= (1\ 3)(2\ 4)(5\ 7)(6\ 9)(8\ 11)(10\ 12), \\ (xy)^{-1} &= (1\ 10\ 6\ 7\ 9\ 11\ 12\ 8\ 5\ 2\ 3\ 4) \end{aligned}$$

- $q = 8$:

$$\begin{aligned} y &= (1\ 3)(2\ 4)(5\ 7)(6\ 8)(9\ 11)(10\ 13)(12\ 15)(14\ 16), \\ (xy)^{-1} &= (1\ 14\ 10\ 11\ 13\ 15\ 16\ 12\ 9\ 6\ 7\ 8\ 5\ 2\ 3\ 4) \end{aligned}$$

- $q = 10$:

$$\begin{aligned} y &= (1\ 3)(2\ 4)(5\ 7)(6\ 8)(9\ 11)(10\ 12)(13\ 15)(14\ 17)(16\ 19)(18\ 20), \\ (xy)^{-1} &= (1\ 18\ 14\ 15\ 17\ 19\ 20\ 16\ 13\ 10\ 11\ 12\ 9\ 6\ 7\ 8\ 5\ 2\ 3\ 4) \end{aligned}$$

These examples illustrate the structure of the construction.

Proof.

(1) In (7.1), the permutation y is written as a product of q transpositions. To show that y has cycle type (2^q) , it suffices to verify that each element of $\{1, \dots, 2q\}$ appears.

For each $s \in \{1, \dots, 2q\}$, we distinguish two cases:

- If $1 \leq s \leq 2q - 8$, let $i = \lfloor (s + 3)/4 \rfloor$. Then $1 \leq i \leq q/2 - 2$, and s appears in α_i as one of $4i - 3$, $4i - 2$, $4i - 1$, or $4i$, according to $s \equiv 1, 2, 3, 0 \pmod{4}$.
- If $2q - 7 \leq s \leq 2q$, then s appears in β .

Thus every element of $\{1, \dots, 2q\}$ appears, and hence y has cycle type (2^q) .

(2) Since xy and $(xy)^{-1}$ have the same cycle type, it suffices to show that xy has cycle type (2^q) . We now prove this.

Table 5 displays a $2q$ -cycle of xy arising from (7.1).

α/β	s	$y \cdot s$	$xy \cdot s$
α_i	$4i - 3$	$4i - 1$	$4i$
	$4i$	$4i - 2$	$4i - 1$
	$4i - 1$	$4i - 3$	$4i - 2$
	$4i - 2$	$4i$	$4i + 1 = 4(i + 1) - 3$
β	$2q - 7$	$2q - 5$	$2q - 4$
	$2q - 4$	$2q - 1$	$2q$
	$2q$	$2q - 2$	$2q - 1$
	$2q - 1$	$2q - 4$	$2q - 3$
	$2q - 3$	$2q - 6$	$2q - 5$
	$2q - 5$	$2q - 7$	$2q - 6$
	$2q - 6$	$2q - 3$	$2q - 2$
	$2q - 2$	$2q$	1

TABLE 5. A $2q$ -cycle of xy ($s = 1, \dots, 2q$)

Under xy , the four elements in the support of α_i are visited in the following order:

$$4i - 3 \rightarrow 4i \rightarrow 4i - 1 \rightarrow 4i - 2 \rightarrow 4(i + 1) - 3 \rightarrow \dots,$$

where the last element $4(i + 1) - 3$ lies in the next block.

If $i < q/2 - 2$, then the next block is α_{i+1} , and the same pattern continues.

If $i = q/2 - 2$, then the next block is β , and since $4(i + 1) - 3 = 2q - 7$, the permutation continues as

$$2q - 7 \rightarrow 2q - 4 \rightarrow 2q \rightarrow 2q - 1 \rightarrow 2q - 3 \rightarrow 2q - 5 \rightarrow 2q - 6 \rightarrow 2q - 2 \rightarrow 1.$$

Thus all $2q$ elements are permuted in a single cycle by xy . Hence xy has cycle type (2^q) .

(3) By Corollary 3.7(2), it suffices to show that

$$\text{for every } 1 \leq k \leq 2q - 1, \quad x^k y \neq y x^k.$$

Equivalently,

$$\text{for every } 1 \leq k \leq 2q - 1,$$

$$\text{there exists } e \in E = \{1, \dots, 2q\} \text{ such that } x^k y \cdot e \neq y x^k \cdot e. \quad (7.2)$$

In what follows, the elements of E are taken modulo $2q$.

(i) Let $e = 2q - 3$. Then

$$\begin{aligned} x^k y \cdot e &= x^k \cdot (2q - 6) = k - 6, \\ y x^k \cdot e &= y \cdot (k - 3). \end{aligned}$$

The equality $x^k y \cdot e = y x^k \cdot e$ holds only if y maps $k - 3$ to $k - 6$, thus decreasing by 3.

From (7.1), the possible values of $k - 3$ are

$$2q - 3, \quad 2q - 1.$$

If $k - 3 = 2q - 3$, then $k = 2q$. Since $1 \leq k \leq 2q - 1$, this case is excluded. Therefore, the equality $x^k y \cdot e = y x^k \cdot e$ can hold only if $k - 3 = 2q - 1$, that is,

$$k = 2. \tag{7.3}$$

(ii) Let $e = 1$. By (7.3), it remains to consider $k = 2$. (7.1) gives

$$y x^k \cdot e = y \cdot (k + 1) = y \cdot 3 = 1.$$

On the other hand,

$$x^k y \cdot e = x^k \cdot 3 = k + 3 = 5.$$

Hence $x^k y \cdot e \neq y x^k \cdot e$. Therefore, the equality $x^k y \cdot e = y x^k \cdot e$ cannot hold for both $e = 2q - 3$ and $e = 1$.

Thus, (7.2) holds. Consequently, for the corresponding dessin d'enfant $\mathcal{D}$, we have $\text{Aut } \mathcal{D} \cong \{1\}$. $\square$

7.2. The Subcase $b \geq 3$

The following proposition was proved in [Ohn26a, Proposition 7.5]. Here we give an alternative proof.

Proposition 7.3. *Every passport $[n, b^q, n]$ of genus ≥ 2 with $b \geq 3$ and $q \geq 2$ admits a dessin with a trivial automorphism group.*

Proof. By (2.4), the genus is

$$\frac{n - (1 + q + 1)}{2} + 1 = \frac{bq - (q + 2)}{2} + 1 = \frac{q(b - 1)}{2}.$$

Since $b \geq 3$ and $q \geq 2$, we have $q(b - 1)/2 \geq 2$. Moreover, since the genus is an integer, if q is odd then b is odd, and hence $n = bq$ is also odd.

By Theorem 3.3, we have

$$\frac{|N|}{|T|} \geq \frac{2}{n + 2} = \frac{2}{n(1 + \frac{2}{n})} \geq \frac{2}{n(1 + \frac{2}{6})} = \frac{3}{2n}.$$

On the other hand, by Proposition 3.1 and Lemma 3.9, we have

$$\frac{|D|}{|T|} \leq \kappa_1 \cdot 2^{\frac{n}{2}} \left(\frac{n}{2}\right)! \cdot \frac{b^q q!}{n!} \quad \left(\left(\frac{n}{2}\right)! := \Gamma\left(\frac{n}{2} + 1\right)\right),$$

where $\kappa_1 = 2623/1894$.

Combining these inequalities, we obtain

$$\frac{|N|}{|D|} \geq \frac{3}{2n} \cdot \frac{n!}{\kappa_1 \cdot 2^{\frac{n}{2}} b^q \left(\frac{n}{2}\right)! q!} = \frac{3}{2bq} \cdot \frac{(bq)!}{\kappa_1 \cdot 2^{\frac{bq}{2}} b^q \left(\frac{bq}{2}\right)! q!}.$$

Let the right-hand side be

$$R(b, q) := \frac{3}{2bq} \cdot \frac{(bq)!}{\kappa_1 \cdot 2^{\frac{bq}{2}} b^q \left(\frac{bq}{2}\right)! q!}. \quad (7.4)$$

The ratio of successive values of $R(b, q)$ with respect to b is given by

$$\begin{aligned} \frac{R(b+1, q)}{R(b, q)} &= \frac{3}{2(b+1)q} \cdot \frac{((b+1)q)!}{\kappa_1 \cdot 2^{\frac{(b+1)q}{2}} (b+1)^q \left(\frac{(b+1)q}{2}\right)! q!} \cdot \frac{2bq}{3} \cdot \frac{\kappa_1 \cdot 2^{\frac{bq}{2}} b^q \left(\frac{bq}{2}\right)! q!}{(bq)!} \\ &= 2^{-\frac{q}{2}} \left(\frac{b}{b+1}\right)^{q+1} \frac{((b+1)q)!}{(bq)!} \cdot \frac{\left(\frac{bq}{2}\right)!}{\left(\frac{(b+1)q}{2}\right)!}. \end{aligned}$$

Here,

$$\frac{((b+1)q)!}{(bq)!} = (bq+1)(bq+2) \cdots (bq+q) > (bq)^q.$$

Moreover, since $q/2 \geq 1$, by Lemma 4.4,

$$\begin{aligned} \frac{\left(\frac{bq}{2}\right)!}{\left(\frac{(b+1)q}{2}\right)!} &= \frac{\Gamma\left(\frac{bq}{2} + 1\right)}{\Gamma\left(\frac{bq}{2} + \frac{q}{2} + 1\right)} \\ &\geq \frac{\Gamma\left(\frac{bq}{2} + 1\right)}{\left(\frac{(b+1)q}{2}\right)^{\frac{q}{2}} \Gamma\left(\frac{bq}{2} + 1\right)} = \left(\frac{2}{(b+1)q}\right)^{\frac{q}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{R(b+1, q)}{R(b, q)} &\geq 2^{-\frac{q}{2}} \left(\frac{b}{b+1}\right)^{q+1} (bq)^q \left(\frac{2}{(b+1)q}\right)^{\frac{q}{2}} \\ &= \frac{b^{2q+1} q^{\frac{q}{2}}}{(b+1)^{\frac{3q}{2}+1}} = \frac{b}{b+1} \left(\frac{b^2 q^{\frac{1}{2}}}{(b+1)^{\frac{3}{2}}}\right)^q. \end{aligned}$$

Since $b \geq 3$ and $q \geq 2$,

$$\frac{b^2 q^{\frac{1}{2}}}{(b+1)^{\frac{3}{2}}} = \frac{b^{\frac{1}{2}} q^{\frac{1}{2}}}{(1 + \frac{1}{b})^{\frac{3}{2}}} \geq \frac{3^{\frac{1}{2}} 2^{\frac{1}{2}}}{(1 + \frac{1}{3})^{\frac{3}{2}}} = \frac{9\sqrt{2}}{8} > 1,$$

therefore we obtain

$$\begin{aligned} \frac{R(b+1, q)}{R(b, q)} &\geq \frac{b}{b+1} \left(\frac{9\sqrt{2}}{8}\right)^q = \frac{1}{1 + \frac{1}{b}} \left(\frac{9\sqrt{2}}{8}\right)^q \\ &\geq \frac{1}{1 + \frac{1}{3}} \left(\frac{9\sqrt{2}}{8}\right)^2 = \frac{243}{128} > 1. \end{aligned}$$

Hence $R(b, q)$ is strictly increasing in b for each q .

The ratio of successive values of $R(b, q)$ with respect to q is given by

$$\begin{aligned} \frac{R(b, q+1)}{R(b, q)} &= \frac{3}{2b(q+1)} \cdot \frac{(b(q+1))!}{\kappa_1 \cdot 2^{\frac{b(q+1)}{2}} b^{q+1} \left(\frac{b(q+1)}{2}\right)! (q+1)!} \cdot \frac{2bq}{3} \cdot \frac{\kappa_1 \cdot 2^{\frac{bq}{2}} b^q \left(\frac{bq}{2}\right)! q!}{(bq)!} \\ &= 2^{-\frac{b}{2}} \frac{q}{b(q+1)^2} \cdot \frac{\left(\frac{bq}{2}\right)!}{\left(\frac{b(q+1)}{2}\right)!} \cdot \frac{(b(q+1))!}{(bq)!}. \end{aligned}$$

Here,

$$\frac{(b(q+1))!}{(bq)!} = (bq+1)(bq+2)\cdots(bq+b) \geq (bq)^b.$$

Moreover, since $b/2 \geq 1$, by Lemma 4.4,

$$\begin{aligned} \frac{\left(\frac{bq}{2}\right)!}{\left(\frac{b(q+1)}{2}\right)!} &= \frac{\Gamma\left(\frac{bq}{2} + 1\right)}{\Gamma\left(\frac{bq}{2} + \frac{b}{2} + 1\right)} \\ &\geq \frac{\Gamma\left(\frac{bq}{2} + 1\right)}{\left(\frac{b(q+1)}{2}\right)^{\frac{b}{2}} \Gamma\left(\frac{bq}{2} + 1\right)} = \left(\frac{2}{b(q+1)}\right)^{\frac{b}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{R(b, q+1)}{R(b, q)} &\geq 2^{-\frac{b}{2}} \frac{q}{b(q+1)^2} (bq)^b \cdot \left(\frac{2}{b(q+1)}\right)^{\frac{b}{2}} \\ &= \left(\frac{q}{q+1}\right)^3 \cdot \left(\frac{b^{\frac{1}{2}}q}{(q+1)^{\frac{1}{2}}}\right)^{b-2}. \end{aligned}$$

When $b = 3$ and $q \geq 3$,

$$\begin{aligned} \frac{R(b, q+1)}{R(b, q)} &\geq \left(\frac{q}{q+1}\right)^3 \cdot \frac{\sqrt{3}q}{(q+1)^{\frac{1}{2}}} = \left(\frac{1}{1+\frac{1}{q}}\right)^3 \cdot \frac{\sqrt{3}q^{\frac{1}{2}}}{\left(1+\frac{1}{q}\right)^{\frac{1}{2}}} \\ &\geq \left(\frac{1}{1+\frac{1}{3}}\right)^3 \cdot \frac{\sqrt{3} \cdot 3^{\frac{1}{2}}}{\left(1+\frac{1}{3}\right)^{\frac{1}{2}}} = \frac{81\sqrt{3}}{128} > 1. \end{aligned}$$

When $b \geq 4$, since $b-2 > 0$ and

$$\frac{b^{\frac{1}{2}}q}{(q+1)^{\frac{1}{2}}} = \left(\frac{bq}{1+\frac{1}{q}}\right)^{\frac{1}{2}} \geq \left(\frac{4 \cdot 2}{1+\frac{1}{2}}\right)^{\frac{1}{2}} = \frac{4}{\sqrt{3}} > 1,$$

we obtain

$$\begin{aligned} \frac{R(b, q+1)}{R(b, q)} &\geq \left(\frac{q}{q+1}\right)^3 \cdot \left(\frac{4}{\sqrt{3}}\right)^{4-2} = \left(\frac{1}{1+\frac{1}{q}}\right)^3 \cdot \frac{16}{3} \\ &\geq \left(\frac{1}{1+\frac{1}{2}}\right)^3 \cdot \frac{16}{3} = \frac{128}{81} > 1. \end{aligned}$$

Hence $R(b, q)$ is strictly increasing in q for all $b \geq 4$, and also for $b = 3$ whenever $q \geq 3$. By direct computation using (7.4), we obtain

$$\begin{aligned} R(3, 2), R(3, 3), R(3, 4), R(4, 2) &\leq 1, \\ R(3, 5), R(4, 4), R(5, 2) &> 1. \end{aligned}$$

Combining these values with the monotonicity of $R(b, q)$ established above, we obtain $R(b, q) > 1$, and hence $|N| > |D|$, for all pairs (b, q) except

$$(b, q) = (3, 2), (3, 3), (3, 4), (4, 2).$$

For these remaining pairs, the values of $|N|$ and $|D|$ are listed in Table 6.

In the case $(b, q) = (3, 3)$, we still have $|N| > |D|$, and therefore a dessin with trivial automorphism group exists.

For the remaining three pairs, we have $|N| < |D|$. For each of them, Table 6 contains an explicit permutation y (with $x = (1\ 2\ \dots\ n)$ fixed) whose corresponding dessin has trivial automorphism group.

Assume that $x = (1\ 2\ \dots\ n)$ is fixed.

b	q	$ N $	$ D $	y with $\text{Aut } \mathcal{D} \cong \{1\}$	$\langle x, y \rangle$
3	2	12	60	$(1\ 2\ 4)(3\ 5\ 6)$	S_5
3	3	464	162		
3	4	38720	47952	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 10)(9\ 11\ 12)$	S_{12}
4	2	276	384	$(1\ 2\ 3\ 5)(4\ 6\ 7\ 8)$	S_8

TABLE 6. Exceptional (b, q) pairs

Therefore, every passport $[n, b^q, n]$ of genus at least 2 with $b \geq 3$ and $q \geq 2$ admits a dessin with trivial automorphism group. $\square$

Remark 7.4. For $b = 2$, the above argument does not apply.

Indeed, when $b = 2$ and the genus is at least 2, we have $n = 2q$ with $q \geq 4$. Since $2 \mid n$, by Proposition 3.6 and Proposition 3.8, we obtain

$$D = \bigcup_{\ell: \text{ prime}, \ell \mid n} D_{\frac{n}{\ell}},$$

$$|D| \geq |D_{\frac{n}{2}}| = |D_q| = 2^q q!.$$

Then, by Proposition 3.1,

$$\frac{|D|}{|T|} \geq 2^q q! \left(\frac{n!}{2^q q!} \right)^{-1} = \frac{2^{2q} (q!)^2}{(2q)!}. \quad (7.5)$$

Moreover,

$$2^{2q} = (1 + 1)^{2q} = \sum_{k=0}^{2q} \binom{2q}{k}.$$

Since all terms in the sum are positive, it follows that

$$2^{2q} > \binom{2q}{q} = \frac{(2q)!}{(q!)^2},$$

hence

$$\frac{2^{2q} (q!)^2}{(2q)!} > 1.$$

Combining this with (7.5), we obtain

$$\frac{|D|}{|T|} > 1.$$

Thus, by the definition (3.1), we have $|N| \leq |T| < |D|$. Therefore, the above method cannot establish $|N| > |D|$. This indicates that a different approach is required for the case $b = 2$.

7.3. Result

Thus we obtain an alternative proof of [Ohn26a, Theorem 7.6].

Theorem 7.5. [Ohn26a, Theorem 7.6] *If a uniform passport $[n, b^q, n]$ ($q \geq 1$) has genus at least 2, then it admits a dessin d'enfant with a trivial automorphism group.*

The same statement holds for the passports $[b^q, n, n]$ and $[n, n, b^q]$.

Proof. Since the automorphism group is symmetric with respect to black vertices, white vertices, and faces, it suffices to prove the statement for the passport $[n, b^q, n]$ ($n = bq$).

By (2.4), the genus is

$$\frac{n - (1 + q + 1)}{2} + 1 = \frac{n - q}{2} = \frac{q(b - 1)}{2} \geq 2.$$

Therefore, we have $b \geq 2$.

The case $b = n$ ($q = 1$) was proved in [Ohn26a, Proposition 7.1].

The case $b = 2$ was proved in Proposition 7.1.

The case $3 \leq b < n$ ($q \geq 2$) was proved in Proposition 7.3. □

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