

修士論文

論文題目

Regularity and Automorphism Groups of
Dessins d'Enfants with Uniform Passports
(一様なパスポートを持つ dessin d'enfant の正則性と自己同型群)

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一様なパスポートを持つ dessin d'enfant の正則性と自己同型群

要旨

代数体上で定義される滑らかな代数曲線に対して、dessin d'enfant と呼ばれる二部グラフを描くことができる。

本論文では、一様なパスポートを持つ dessin d'enfant、すなわち黒頂点・白頂点・面の分岐数がそれぞれ一定なものについて、正則性と自己同型群がどのように分布するか、それらが種数によってどのように変化するかを調べた。一様なデッサンは高い対称性を持つが必ずしも正則であるとは限らないというところに着目した。

主な成果は、以下を示したことである。

- (1) $[a^p, b^q, n]$ の形のパスポート (木の場合) は、 $\gcd(p, q) = 1$ のとき、かつそのときに限り正則なデッサンを持つ
- (2) $[n, b^q, n]$ の形の種数 2 以上のパスポートは自己同型群が自明なデッサンを持つ

加えて、種数 0, 1 の一様なパスポートについてのいくつかの命題を証明した。

さらに、自己同型群を調べるために有用な、対称群の元の数え上げに関する 2 つの定理の証明を行った。

Cette découverte, qui techniquement se réduit à si peu de choses, a fait sur moi une impression très forte, et elle représente un tournant décisif dans le cours de mes réflexions, un déplacement notamment de mon centre d'intérêt en mathématique, qui soudain s'est trouvé fortement localisé. Je ne crois pas qu'un fait mathématique m'ait jamais autant frappé que celui-là, et ait eu un impact psychologique comparable.

この発見は、技術的にはごくわずかな事柄に還元されるにもかかわらず、私にきわめて強い印象を与えた。そして私の思索の歩みにおける決定的な転換点となり、とりわけ数学における興味の中心の変化をもたらし、突如としてそこに強く集中することになった。数学的事実がこれほどまでに私に強い衝撃を与えたことも、これに匹敵する心理的な影響を及ぼしたことも、かつてなかったのである。

— Alexandre Grothendieck, *Esquisse d'un Programme* (1984)

REGULARITY AND AUTOMORPHISM GROUPS OF DESSINS D'ENFANTS WITH UNIFORM PASSPORTS

TATSUYA OHNISHI

ABSTRACT. For a smooth algebraic curve defined over a number field, one can associate a bipartite graph known as a *dessin d'enfant*.

In this paper, we investigate the regularity and automorphism groups of dessins d'enfants with *uniform passports*, that is, those for which the valencies of black vertices, white vertices, and faces are constant, and study how these properties depend on the genus. Although uniformity imposes a high degree of symmetry, such dessins are not necessarily regular.

Our main results are as follows: (1) A passport of the form $[a^p, b^q, n]$ (the tree case) admits a regular dessin if and only if $\gcd(p, q) = 1$. (2) Every passport of the form $[n, b^q, n]$ of genus at least 2 admits a dessin with a trivial automorphism group.

In addition, we obtain several results on uniform passports of genus 0 and 1. We also establish two theorems on the enumeration of elements in symmetric groups, which are useful for the study of automorphism groups of dessins.

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1. Introduction

1.1. Belyi's Theorem and Dessins d'Enfants

Theorem 1.1 (Belyi's Theorem). [Bel79][Bel02][JW16, Theorem 1.3] *Let X be a compact Riemann surface, that is, a smooth projective algebraic curve in the complex projective space $\mathbb{P}_{\mathbb{C}}^N$ for some N . Then X can be defined over the field of algebraic numbers $\overline{\mathbb{Q}}$ if and only if there exists a non-constant meromorphic function $\beta: X \rightarrow \widehat{\mathbb{C}} (:= \mathbb{C} \cup \{\infty\})$ ramified over at most three points.*

Everything begins with this striking and deep theorem. As a consequence of this result, for any compact Riemann surface defined over a number field, one can choose a suitable function β (called a *Belyi function*) such that all its critical values lie within three points.

By applying an appropriate Möbius transformation, these three points can be taken to be 0, 1, and ∞ . For such a pair (X, β) (called a *Belyi pair*), one can draw a bipartite graph called a *dessin d'enfant* (child's drawing), or simply a *dessin* (see Definition 2.1).

The dessin for (X, β) is drawn on an orientable surface of the same genus as X , where the black vertices ($\bullet$) and white vertices ($\circ$) represent $\beta^{-1}(0)$ and $\beta^{-1}(1)$, respectively¹. The faces — that is, the connected components bounded by edges — correspond to $\beta^{-1}(\infty)$, and the edges correspond to $\beta^{-1}([0, 1])$. Each face is homeomorphic to a disk.

By studying the properties of dessins d'enfants, one can combine insights from algebraic geometry and combinatorics, including graph theory, to enrich both areas of analysis. This perspective also opens up a range of possibilities for further applications.

Two examples of dessins d'enfants are shown in Figure 1. In the left example, $X = \widehat{\mathbb{C}}$ (of genus 0) and $\beta = -z^4(3z - 5)/(5z - 3)$, hence the dessin is drawn on a sphere. The preimage $\beta^{-1}(0)$ consists of the points 0, $5/3$ with valencies 4 and 1, respectively; $\beta^{-1}(1)$ consists of the points 1, $(-2 \pm \sqrt{5}i)/3$ with valencies 3, 1, and 1; and $\beta^{-1}(\infty)$ consists of the points $3/5, \infty$ with valencies 1 and 4.

In the right example, X is the curve defined by $x^3 + y^3 = z^3$ (the Fermat curve of degree 3 and genus 1) in $\mathbb{P}_{\mathbb{C}}^2$, and $\beta = x^3/z^3$. Let ζ_3 denote a primitive cube root of unity. This dessin has three faces corresponding to the points $[1, -1, 0]$, $[1, -\zeta_3, 0]$, and $[1, -\zeta_3^2, 0]$.

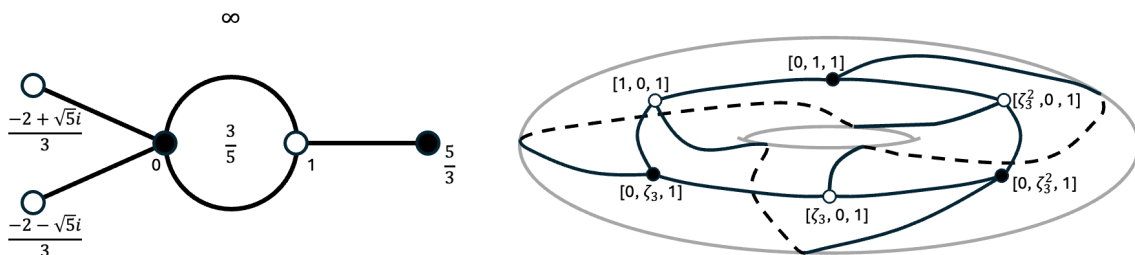


FIGURE 1. Examples of dessins d'enfants

The number of edges of a dessin coincides with the degree of the Belyi function. Note that the degree of a Belyi function is defined as the number of points in the preimage of a non-critical value; this does not necessarily coincide with the degree of β when it is regarded as a rational function. In fact, in the right example above, the degree of the Belyi function is 9, whereas the degree of β as a rational function is 3.

¹ Some references (such as [JW16]) represent $\beta^{-1}(0)$ by white vertices and $\beta^{-1}(1)$ by black vertices. In this paper, we adopt the convention used in many classical references.

Two dessins $\mathcal{D}$ and $\mathcal{D}'$, corresponding respectively to the Belyĭ pairs (X, β) and (X', β') , are said to be *isomorphic* if there exists a homeomorphism $\varphi: X \rightarrow X'$ such that $\beta' \circ \varphi = \beta$. Equivalently, $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic if there exists an orientation-preserving homeomorphism between X and X' that induces an isomorphism of the embedded bipartite graphs, preserving vertex colors and the cyclic order of incident edges at each vertex.

For a dessin d'enfant, one can define an action of the absolute Galois group $\mathbb{G} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ [JW16, 4.2.1]. Given a dessin $\mathcal{D}$ associated with a Belyĭ pair (X, β) and an element $\sigma \in \mathbb{G}$, we denote by (X^σ, β^σ) the pair obtained by the action of σ on X and β . We then denote by $\mathcal{D}^\sigma$ the dessin corresponding to the Belyĭ pair (X^σ, β^σ) .

Remark 1.2. The Belyĭ pair (X^σ, β^σ) is obtained as follows.

Suppose that a compact Riemann surface X is defined over a number field by polynomial equations of the form

$$X = \{[x_0, \dots, x_N] \in \mathbb{P}_{\mathbb{C}}^N \mid f_1(x_0, \dots, x_N) = \dots = f_m(x_0, \dots, x_N) = 0\},$$

$$f_i \in \overline{\mathbb{Q}}[X_0, \dots, X_N].$$

For $\sigma \in \mathbb{G}$ and each polynomial f_i , let f_i^σ denote the polynomial obtained by applying σ to all coefficients of f_i . We then define the curve obtained by the action of σ on X as

$$X^\sigma = \{[x_0, \dots, x_N] \in \mathbb{P}_{\mathbb{C}}^N \mid f_1^\sigma(x_0, \dots, x_N) = \dots = f_m^\sigma(x_0, \dots, x_N) = 0\}. \quad (1.1)$$

Since σ is an automorphism of $\widehat{\mathbb{C}}$ (with $\sigma(\infty) = \infty$), we have

$$\begin{aligned} [y_0, \dots, y_N] &\in X^\sigma \\ &\Leftrightarrow f_1^\sigma(y_0, \dots, y_N) = \dots = f_m^\sigma(y_0, \dots, y_N) = 0 \\ &\Leftrightarrow \sigma^{-1}(f_1^\sigma(y_0, \dots, y_N)) = \dots = \sigma^{-1}(f_m^\sigma(y_0, \dots, y_N)) = 0 \\ &\Leftrightarrow (f_1^\sigma)^{\sigma^{-1}}(\sigma^{-1}(y_0), \dots, \sigma^{-1}(y_N)) = \dots = (f_m^\sigma)^{\sigma^{-1}}(\sigma^{-1}(y_0), \dots, \sigma^{-1}(y_N)) = 0 \\ &\Leftrightarrow f_1(\sigma^{-1}(y_0), \dots, \sigma^{-1}(y_N)) = \dots = f_m(\sigma^{-1}(y_0), \dots, \sigma^{-1}(y_N)) = 0 \\ &\Leftrightarrow [\sigma^{-1}(y_0), \dots, \sigma^{-1}(y_N)] \in X \\ &\Leftrightarrow \text{there exists } [x_0, \dots, x_N] \in X \text{ such that } [y_0, \dots, y_N] = [\sigma(x_0), \dots, \sigma(x_N)] \\ &\Leftrightarrow [y_0, \dots, y_N] \in \{[\sigma(x_0), \dots, \sigma(x_N)] \mid [x_0, \dots, x_N] \in X\}. \end{aligned}$$

Therefore, an equivalent formulation of (1.1) is

$$X^\sigma = \{[\sigma(x_0), \dots, \sigma(x_N)] \mid [x_0, \dots, x_N] \in X\}.$$

Let β be a Belyĭ function on X . We define β^σ in the same way as f_i^σ , namely by applying σ to all coefficients of β . Since $\sigma(0) = 0$, $\sigma(1) = 1$, and $\sigma(\infty) = \infty$ for any $\sigma \in \mathbb{G}$, the map β^σ is again a Belyĭ function on X^σ .

Therefore, for a Belyĭ pair (X, β) , the pair (X^σ, β^σ) is also a Belyĭ pair.

In this way, one obtains an action of an element $\sigma \in \mathbb{G}$ on a dessin d'enfant $\mathcal{D}$. Although $\mathcal{D}^\sigma$ has the same number of edges and the same *passport* (i.e. the same list of ramification indices) as $\mathcal{D}$, the incidence relations between vertices may change. Consequently, $\mathcal{D}^\sigma$ need not be isomorphic to $\mathcal{D}$ as a dessin.

The action of $\mathbb{G}$ on the set of all dessins is faithful; that is, for any two distinct elements of $\mathbb{G}$, there exists a dessin whose images under these elements lie in distinct isomorphism classes. Therefore, the study of Galois orbits of dessins provides a powerful tool for investigating the structure of the absolute Galois group.

1.2. Regularity and Automorphism Groups

Two fundamental invariants (up to isomorphism) of a dessin d'enfant with respect to the Galois action are its regularity and its automorphism group. A dessin is said to be *regular* if its *monodromy group* (see Definition 2.6) acts regularly (that is, freely and transitively) on the set of edges. From a geometric point of view, the *automorphism group* of a dessin is the group of deck transformations of the associated Belyĭ covering. Equivalently, it can be identified with the centralizer of the monodromy group in the symmetric group acting on the edges.

By studying regularity and automorphism groups, one can gain insight into the symmetry properties of a dessin d'enfant. The order of the automorphism group always divides the number of edges of the dessin, and the equality of these two numbers is equivalent to the dessin being regular. Thus, the structure of the automorphism group provides a precise measure of the symmetry exhibited by the dessin.

Figure 2 shows two dessins with the same passport; that is, they have the same valency list: each has 8 edges, two black vertices of valency 4, four white vertices of valency 2, and two faces of valency 4. Although both dessins may appear highly symmetric at first glance, the left dessin is regular, whereas the right one is not. The automorphism group of the left dessin has order 8, while that of the right dessin has order 4.

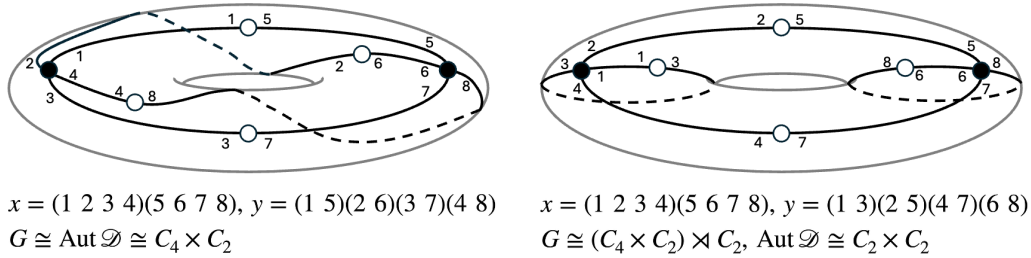


FIGURE 2. Regular and non-regular dessins with the same passport

Thus, regularity and automorphism groups may differ among dessins with the same passport. However, they are invariant under the action of the absolute Galois group. They therefore play an essential role in the study of families of dessins lying in the same Galois orbit.

Furthermore, regularity and automorphism groups have a significant impact on the relationship between the field of moduli and the field of definition.

For a dessin d'enfant $\mathcal{D}$ corresponding to a Belyĭ pair (X, β) , the *field of moduli* $M(\mathcal{D})$ is defined as the fixed field of the subgroup $G(\mathcal{D}) = \{\sigma \in \mathbb{G} \mid \mathcal{D} \cong \mathcal{D}^\sigma\} \leq \mathbb{G}$. That is, $M(\mathcal{D})$ is the subfield of $\overline{\mathbb{Q}}$ consisting of those elements that are fixed by every automorphism σ for which the conjugate dessin $\mathcal{D}^\sigma$ is isomorphic to $\mathcal{D}$.

A number field K is called a *field of definition* of $\mathcal{D}$ if both X and β can be defined over K . Unlike the field of moduli, a field of definition of $\mathcal{D}$ is in general not unique, and there need not exist a smallest field of definition.

The field of moduli depends only on the isomorphism class of $\mathcal{D}$ and is contained in any field of definition of $\mathcal{D}$. For most dessins d'enfants, the field of moduli is also a field of definition; that is, X and β can be defined over $M(\mathcal{D})$. However, this is not always the case. In such situations, there is no field of definition of $\mathcal{D}$ that is fixed by all $\sigma \in G(\mathcal{D})$.

In the study of dessins d'enfants, the relationship between the field of moduli and fields of definition is strongly influenced by both the genus and the combinatorial structure of the dessin. If a dessin has a trivial automorphism group, then its field of moduli is

necessarily a field of definition, independently of the genus or the passport. This follows from rigidity of descent, which forces the associated Weil cocycle to be trivial.

The same conclusion holds for regular dessins, that is, for dessins whose associated Belyi functions are Galois coverings. In this case, the large symmetry encoded in the automorphism group allows the descent data to be normalized, so that the dessin descends to its field of moduli. Consequently, regular dessins do not give rise to genuine descent obstructions.

By contrast, when a dessin has a nontrivial automorphism group but is not regular, the behavior of the field of moduli becomes more subtle. As shown by Cueto [Cue14], in this intermediate situation the field of moduli need not be a field of definition. From a cohomological viewpoint, the obstruction to descent is measured by a Weil cocycle with values in the automorphism group, and explicit examples arise precisely when this cocycle represents a nontrivial class.

In this paper, we develop methods for analyzing how regular dessins and dessins with trivial automorphism groups are distributed as the genus varies. These results contribute to a deeper understanding of the relationship between the field of moduli and the field of definition.

1.3. Uniform Passports and Dessins

A dessin d'enfant is said to be *uniform* if the valencies of black vertices, white vertices, and faces are each constant. One also says that it has a uniform passport (or uniform valency list).

Uniform dessins exhibit a high degree of symmetry. However, while every regular dessin is uniform, the converse does not hold: a uniform dessin need not be regular. In other words, uniformity does not represent the highest possible level of symmetry.

As an example, consider uniform passports $[a^p, b^q, c^r]$, where $n = pa = bq = rc$, and suppose that $n = 6$. By symmetry among black vertices, white vertices, and faces, we may assume $c \geq a \geq b$ (equivalently, $r \leq p \leq q$).

In genus 0, the uniform passports are $[6, 1^6, 6]$ and $[2^3, 2^3, 3^2]$. The corresponding dessins are shown in Figure 3. In both cases, the dessins are regular.

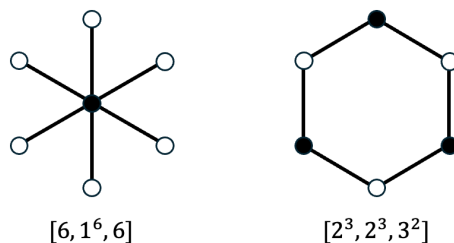


FIGURE 3. Uniform dessins d'enfants of genus 0, degree 6

In genus 1, the uniform passports are $[3^2, 2^3, 6]$ and $[3^2, 3^2, 3^2]$, and the corresponding dessins are shown in Figure 4. The dessin on the left is regular, whereas the one on the right is not. The automorphism group of the right-hand dessin has order 2, which is strictly smaller than 6.

For genus at least 2, the only uniform passport is $[6, 3^2, 6]$, which has genus 2. There are four dessins with this passport, shown in Figure 5. The orders of their automorphism groups are 6, 3, 2, and 1, respectively, from the upper left to the lower right. Only the upper-left dessin is regular; the others are non-regular, and the lower-right dessin has a trivial automorphism group.

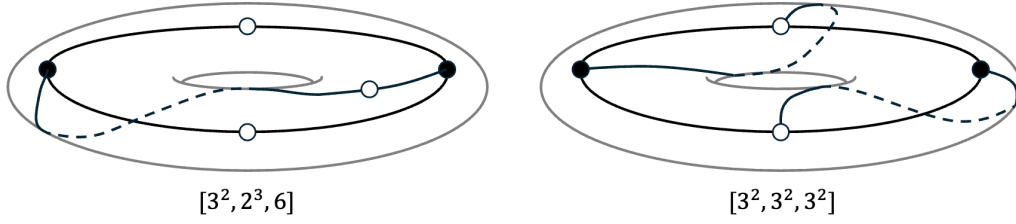


FIGURE 4. Uniform dessins d'enfants of genus 1, degree 6

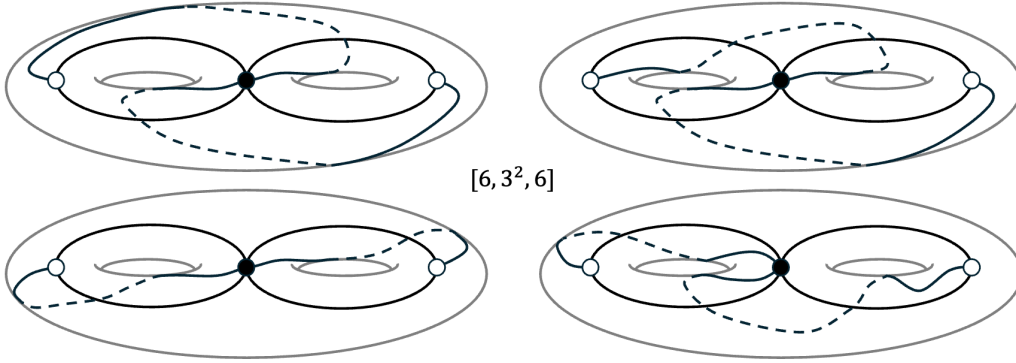


FIGURE 5. Uniform dessins d'enfants of genus 2, degree 6

1.4. Related Work

The study of regularity and automorphism groups of dessins d'enfants has been approached from several complementary perspectives, including group-theoretic constructions, topological realization results, and arithmetic considerations, particularly in the presence of uniform or highly symmetric combinatorial data. We briefly recall some relevant results in these directions.

Jones [Jon14] shows how, for a finite group G , regular dessins with automorphism group G can be enumerated, represented as quotients of a single regular dessin $U(G)$, and analyzed under the action of certain hypermap operations. Examples for several classes of groups, such as the cyclic group C_n , the dihedral group D_n , the symmetric group S_n , the alternating group A_n , and certain non-abelian finite simple groups, are provided.

Complementarily, Hidalgo [Hid19] proves that for every possible action of a finite group G by orientation-preserving homeomorphisms on a closed orientable surface of genus $g \geq 2$, there exists a dessin whose automorphism group is isomorphic to G and which realizes the corresponding topological action of G .

Girondo, Torres-Teigell, and Wolfart [GTTW14] investigate the arithmetic and moduli-theoretic properties of uniform dessins on quasiplatonic surfaces, that is, curves admitting a regular dessin. In the arithmetic quasiplatonic setting they consider, they show that the field of moduli of a quasiplatonic curve is remarkably stable: not only does the curve itself descend to its field of moduli, but the same field also serves as the field of moduli for all uniform non-regular dessins arising from non-normal inclusions of triangle groups on that curve. Moreover, under mild additional geometric conditions, these dessins can in fact be defined over this field.

Horie [Hor24] studies the enumeration of equivalence classes of dessins whose automorphism groups have a prescribed order r . The focus is on dessins with two vertices, which necessarily have passports (valency lists) of the form $[n^1, n^1, \lambda]$ with $\lambda \vdash n$. Explicit enumeration formulas are obtained for two cases: (1) dessins with L faces, that is, with $l(\lambda) = L$; and (2) dessins with h faces of degree 2.

Taken together, these works demonstrate that regular dessins and their automorphism groups are highly constrained and well understood from both group-theoretic and topological viewpoints, while uniformity alone does not control regularity, descent, or symmetry.

1.5. Purpose and Main Results

By computing automorphism groups of uniform dessins d'enfants and examining their regularity, one observes a tendency for the proportion of regular dessins to decrease as the genus increases.

This naturally leads to the following questions:

For a given uniform passport,

- Under what conditions do regular dessins d'enfants exist, and how many are there?
- How are the automorphism groups distributed?
- How does this distribution change with the genus?

In response to these questions, we expect the situation to be as summarized in Table 1, where n denotes the number of edges of the dessin.

Genus	$\text{Aut } \mathcal{D} \cong \{1\}$	$1 < \text{Aut } \mathcal{D} < n$	$ \text{Aut } \mathcal{D} = n$ (regular)
0	– ⁽¹⁾	– ⁽¹⁾	✓ ⁽¹⁾
1 (with an (n) -cycle)	– ⁽³⁾	– ⁽²⁾	✓ ⁽⁵⁾
1 (without an (n) -cycle)	– ⁽³⁾	✓ ⁽²⁾	◇
≥ 2 (with ≥ 2 (n) -cycles)	✓ ⁽⁴⁾	◇	✓ ⁽⁵⁾
≥ 2 (with ≤ 1 (n) -cycle)	✓ [*]	✓ [*]	◇

✓: always occurs, –: never occurs, ◇: depends on the passport.

TABLE 1. Distribution of automorphism groups for uniform passports

Asterisks (*) indicate statements that are conjectural at the present stage. The numbers in parentheses refer to the corresponding items listed below.

- (1) Genus 0: all uniform dessins are regular and have automorphism groups of order n , isomorphic to their monodromy groups.
- (2) Genus 1: a uniform passport admits a non-regular dessin if and only if it does not contain an element of type n^1 (that is, it does not correspond to a tree).
- (3) Genus 1: no uniform passport admits a dessin with a trivial automorphism group.
- (4) Genus ≥ 2 : every uniform passport admits a dessin with a trivial automorphism group, and hence admits a non-regular dessin. Moreover, if the passport contains at most one cycle of type (n) , then it also admits a non-regular dessin with a nontrivial automorphism group.
- (5) Any genus: a passport of the form $[a^p, b^q, n]$, or any permutation thereof in the tree case, admits a regular dessin if and only if $\gcd(p, q) = 1$.

The main contribution of this paper is to provide proofs, in the following form, for statement (5) and for certain cases of statement (4).

Theorem 6.1. *For a uniform passport $[a^p, b^q, n]$ ($p, q \geq 1$), there exists a regular dessin d'enfant with this passport if and only if $\gcd(p, q) = 1$.*

The same statement holds for the passports $[a^p, n, b^q]$ and $[n, a^p, b^q]$.

Theorem 7.6. *If a uniform passport $[n, b^q, n]$ ($q \geq 1$) has genus at least 2, then it admits a dessin d'enfant with a trivial automorphism group.*

The same statement holds for the passports $[b^q, n, n]$ and $[n, n, b^q]$.

For the proof of Theorem 7.6, we make use of the relationship between the primitivity of the monodromy group of a dessin and the triviality of its automorphism group. This approach is useful for investigating the distribution of dessins with low symmetry.

From the perspective of the field of moduli and the field of definition, the above results broaden the range of cases in which the field of moduli can be shown to be a field of definition. At the same time, they help to narrow down and identify the cases in which this coincidence fails.

In addition, we provide proofs of (1) (**Proposition 4.1**), (2) (**Proposition 5.1**), and (3) (**Proposition 5.2**), as well as a proposition concerning regularity based on properties of abstract groups (**Proposition 6.3**). Although the proofs of these propositions are straightforward, we could not find them explicitly stated in the literature. For completeness, we therefore provide proofs here.

We also establish the following two theorems concerning enumeration in symmetric groups, which are needed to determine the distribution of dessins with trivial automorphism group.

In the following theorem, let $T(b, q)$ denote the number of permutations in the symmetric group S_n with cycle type (b^q) , and let $N(b, q)$ denote the number of permutations $y \in S_n$ of cycle type (b^q) such that xy has cycle type (n) .

Theorem 3.3. *Let $n \geq 3$ be an integer such that $n = bq$, with $b \geq 2$, and assume that $n \equiv q \pmod{2}$. Fix $x = (1\ 2\ \dots\ n) \in S_n$. Then*

$$\frac{N(b, q)}{T(b, q)} \geq \frac{2}{n + 2}.$$

In particular, the inequality becomes an equality when $b = 2$.

Theorem 3.5. *Fix $x = (1\ 2\ \dots\ n) \in S_n$. Among the elements $y \in S_n$ of cycle type (b^q) , the number of those for which $G = \langle x, y \rangle$ becomes an imprimitive group with respect to residue classes modulo m ($m \mid n$, $2 \leq m < n$) is given by*

$$I_m(b, q) = \frac{m!}{b^q} \left(\left(\frac{n}{m} \right)! \right)^m \sum_{\{(d_i, t_i)\}} \prod_i \frac{1}{d_i^{t_i} t_i!} \left(\frac{d_i^{\frac{d_i q}{m}}}{\left(\frac{d_i q}{m} \right)!} \right)^{t_i},$$

where the summation on the right-hand side is taken over all partitions $\{(d_i, t_i)\}$ of m satisfying

$$\sum_i d_i t_i = m, \quad d_i \text{ are all distinct,} \quad d_i \mid b, \quad m \mid d_i q \text{ for all } i.$$

For future work, two directions remain. First, to establish the conjectural statements summarized in Table 1. This includes, in particular, proving the existence of dessins with trivial automorphism groups for other uniform passports of genus at least 2. Second, to obtain more quantitative results on regularity and automorphism groups in general.

2. Preliminaries

The definitions and results in this chapter are based primarily on [JW16] and [APZ20].

Definition 2.1 (Dessin d'enfant). [JW16, Definition 2] A *dessin d'enfant*, or simply a *dessin*, is a map consisting of a connected, finite, bipartite graph embedded in a connected, compact, oriented surface without boundary. Here, a bipartite graph is a graph whose vertices can be colored black and white in such a way that each edge joins a black vertex to a white vertex.

Since a compact Riemann surface provides a suitable surface on which a dessin d'enfant can be embedded, one can draw a dessin corresponding to a Belyĭ pair (X, β) , as described in Section 1.1.

There is another convention for dessins d'enfants. In this “tripartite” convention, the points of $\beta^{-1}(\infty)$ are represented by red vertices instead of faces, and the edges correspond to the preimages of the intervals $[0, 1]$, $[\infty, 0]$, and $[1, \infty]$. Each face is homeomorphic to a disk and is incident to three edges.

Figure 6 shows the tripartite and bipartite graphs corresponding to the left example in Figure 1 (gray vertices are used instead of red).

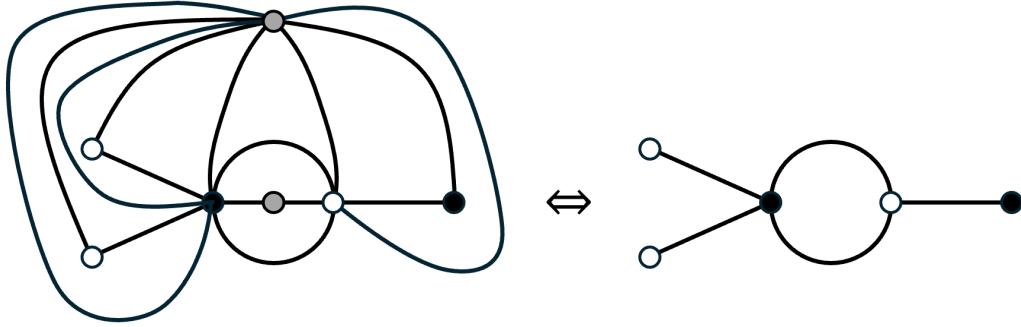


FIGURE 6. An example of a dessin d'enfant in the tripartite and bipartite conventions

At first glance, the bipartite graph may appear to contain less information; however, under the defining properties of a Belyĭ function, the following proposition establishes the equivalence of the two conventions.

Proposition 2.2. *For a Belyĭ function β , the following two types of dessins d'enfants are in one-to-one correspondence.*

- (1) *The dessin in which the points of $\beta^{-1}(0)$, $\beta^{-1}(1)$, and $\beta^{-1}(\infty)$ are represented by black, white, and red vertices, respectively, and the preimages $\beta^{-1}([0, 1])$, $\beta^{-1}([\infty, 0])$, and $\beta^{-1}([1, \infty])$ are represented by edges.*
- (2) *The dessin obtained from (1) by deleting the red vertices together with the edges incident to them, and by regarding the faces as the preimage $\beta^{-1}(\infty)$; as a result, the edges represent $\beta^{-1}([0, 1])$.*

Proof. The correspondence from (1) to (2) is straightforward: the dessin in (2) is obtained by simply deleting the red vertices together with the edges incident to them, which correspond to $\beta^{-1}([\infty, 0])$ and $\beta^{-1}([1, \infty])$.

Conversely, consider a dessin drawn according to convention (2). Take one of its faces F , and let v denote the number of black and white vertices incident to F , e_1 the number of edges contained only in F , and e_2 the number of edges shared with other faces. Then F has v vertices, $e_1 + e_2$ edges, and consists of exactly one face.

On $\widehat{\mathbb{C}}$, if we regard 0, 1, and ∞ as vertices and $[0, 1]$, $[\infty, 0]$, and $[1, \infty]$ as edges, then $\widehat{\mathbb{C}}$ is subdivided into two triangles. Since the Belyĭ function β is a non-constant meromorphic function, the pullback under β preserves this triangulation. Therefore, a dessin according to convention (1) consists of triangles, and each triangle has one black vertex, one white vertex, and one red vertex.

When the above face F is drawn according to convention (1), let r denote the number of red vertices. In this case, F has $v + r$ vertices.

By triangulation, each of the e_1 edges gives rise to two triangles, since both sides of each such edge lie inside F ; hence four edges are added to connect it to red vertices. On the other hand, each of the e_2 edges gives rise to one triangle in F , and two edges are added.

Since each edge is shared by two triangles, the total number of edges in F is

$$e_1 + e_2 + \frac{4e_1 + 2e_2}{2} = 3e_1 + 2e_2.$$

Since two triangles are created for each of the e_1 edges and one triangle for each of the e_2 edges, the total number of faces in F is $2e_1 + e_2$.

The Euler characteristic (number of vertices + number of faces – number of edges) of F is invariant under triangulation. Hence we have

$$v + 1 - (e_1 + e_2) = (v + r) + (2e_1 + e_2) - (3e_1 + 2e_2),$$

which implies that $r = 1$.

Therefore, F always has exactly one red vertex. Consequently, a dessin drawn according to convention (2) corresponds uniquely to a dessin drawn according to convention (1), with each face regarded as a point of $\beta^{-1}(\infty)$. $\square$

In what follows, we adopt the bipartite convention, which involves fewer vertices and edges.

A *partition* λ of a positive integer n , denoted by $\lambda \vdash n$, is a multiset of positive integers whose sum is n , where the order of the parts is irrelevant.

Definition 2.3 (Passport of a dessin). [APZ20, Definition 2.10] Let n be the number of edges of a dessin d'enfant. The triple $[\lambda_0, \lambda_1, \lambda_\infty]$ of partitions $\lambda_0, \lambda_1, \lambda_\infty \vdash n$, which correspond respectively to the valencies of the black vertices ($\beta^{-1}(0)$), the white vertices ($\beta^{-1}(1)$), and the faces ($\beta^{-1}(\infty)$) of the dessin, is called a *passport* of the dessin.

The dessins in Figure 1 have passports $[41, 311, 41]$ and $[3^3, 3^3, 3^3]$, respectively. Here the symbols 41, 311, and 3^3 denote the partitions (4, 1), (3, 1, 1), and (3, 3, 3), respectively.

Since $\lambda_0, \lambda_1, \lambda_\infty$ are partitions of n , we have

$$|\lambda_0| = |\lambda_1| = |\lambda_\infty| = n, \tag{2.1}$$

where $|\lambda|$ denotes the sum of the parts of the partition λ .

The numbers of vertices, faces, and edges are $l(\lambda_0) + l(\lambda_1)$, $l(\lambda_\infty)$, and n , respectively, where $l(\lambda)$ denotes the number of parts of the partition λ . Therefore, the genus g of the underlying curve X satisfies

$$l(\lambda_0) + l(\lambda_1) + l(\lambda_\infty) - n = 2 - 2g,$$

and hence

$$g = \frac{n - (l(\lambda_0) + l(\lambda_1) + l(\lambda_\infty))}{2} + 1.$$

Since g is a non-negative integer, it follows that

$$l(\lambda_0) + l(\lambda_1) + l(\lambda_\infty) \leq n + 2, \quad l(\lambda_0) + l(\lambda_1) + l(\lambda_\infty) \equiv n \pmod{2}. \quad (2.2)$$

Note that not every triple of partitions satisfying (2.1) and (2.2) necessarily corresponds to a dessin d'enfant. For example, although $[2^2, 2^2, 31]$ formally satisfies these conditions and would yield genus 0, there exists no dessin with this passport. Similarly, the passport $[3^2, 3^2, 42]$, which would correspond to genus 1, admits no dessin.

These facts can be proved by showing that there exists no corresponding monodromy group (see Definition 2.6) in each case.

Definition 2.4 (Uniform passports and dessins). [JW16, Remark 3.2] The passport of a dessin d'enfant given by $[\lambda_0, \lambda_1, \lambda_\infty] = [a_1 \cdots a_p, b_1 \cdots b_q, c_1 \cdots c_r]$ is called *uniform* if

$$a_1 = \cdots = a_p, \quad b_1 = \cdots = b_q, \quad c_1 = \cdots = c_r,$$

i.e. if the passport takes the form $[a^p, b^q, c^r]$.

A dessin d'enfant is called *uniform* if it has a uniform passport.

For a uniform passport $[a^p, b^q, c^r]$ with $n = pa = qb = rc$ and the genus g of the underlying curve X corresponding to a dessin with this passport, by (2.1) we have

$$p + q + r - n = 2 - 2g, \quad (2.3)$$

and hence

$$g = \frac{n - (p + q + r)}{2} + 1. \quad (2.4)$$

Proposition 2.5. *For each $g \geq 2$, there are only finitely many uniform passports of genus g , and the maximum degree is $84(g - 1)$.*

Proof. It follows from (2.3) that

$$p + q + r = n - 2(g - 1).$$

Since $n = pa = qb = rc$, we obtain

$$\frac{n}{a} + \frac{n}{b} + \frac{n}{c} = n - 2(g - 1).$$

When $g \geq 2$, this implies

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 - \frac{2(g - 1)}{n} < 1. \quad (2.5)$$

Since this expression is symmetric in a, b , and c , we may assume $a \leq b \leq c$ (and hence $p \geq q \geq r$) when discussing the finiteness of passports for fixed genus g and an upper bound for the degree n . Under this assumption, we obtain

$$\begin{aligned} \frac{1}{a} &< 1, \text{ hence } a \geq 2, \\ \frac{2}{b} &\leq \frac{1}{a} + \frac{1}{b} < 1, \text{ hence } b \geq 3. \end{aligned}$$

Let $\alpha = 1 - (1/a + 1/b + 1/c)$. Then $0 < \alpha < 1$, and from (2.5),

$$n = \frac{2(g - 1)}{\alpha}. \quad (2.6)$$

Hence, with g fixed, n is decreasing in α .

(i) When $a = 2$ and $b = 3$.

In this case,

$$\alpha = 1 - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{c} \right) = \frac{1}{6} - \frac{1}{c}.$$

Since $\alpha > 0$, we have $c \geq 7$. Therefore,

$$\alpha \geq \frac{1}{6} - \frac{1}{7} = \frac{1}{42}.$$

By (2.6) we obtain

$$n \leq \frac{2(g-1)}{1/42} = 84(g-1),$$

and equality is attained when $c = 7$.

(ii) When $a = 2$ and $b \geq 4$.

In this case, we have $c \geq 4$ since $c \geq b$. However, the case $b = c = 4$ is excluded since $(a, b, c) = (2, 4, 4)$ does not satisfy (2.5). Hence we have $c \geq 5$, and therefore,

$$\alpha \geq 1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{5} \right) = \frac{1}{20}.$$

By (2.6) we obtain

$$n \leq \frac{2(g-1)}{1/20} = 40(g-1).$$

(iii) When $a \geq 3$.

In this case, we have $b, c \geq 3$ since $b, c \geq a$. However, $a = b = c = 3$ is excluded since $(a, b, c) = (3, 3, 3)$ does not satisfy (2.5). Hence we have $c \geq 4$, and thus

$$\alpha \geq 1 - \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{4} \right) = \frac{1}{12}.$$

By (2.6) we obtain

$$n \leq \frac{2(g-1)}{1/12} = 24(g-1).$$

From cases (i)–(iii), we conclude that for any fixed genus $g \geq 2$, the maximum possible value of n is $84(g-1)$. This bound is attained when $(a, b, c) = (2, 3, 7)$.

Therefore, there are finitely many values of n for each g . Since a , b , and c divide n , there are only finitely many possible choices for each of them for a given n . Hence, for each genus $g \geq 2$, there exist only finitely many uniform passports. $\square$

Definition 2.6 (Monodromy group of a dessin d'enfant). [JW16, 2.1.1] Define two permutations x and y acting on the set of edges E of a dessin d'enfant $\mathcal{D}$ as follows. For each edge $e \in E$, define $x \cdot e$ and $y \cdot e$ to be the next edges around the unique black vertex and the unique white vertex incident to e , respectively, following the counterclockwise orientation.

The *monodromy group* of $\mathcal{D}$ is the subgroup $G = \langle x, y \rangle$ generated by x and y in the symmetric group $\text{Sym}(E)$ of all permutations of E .

Since a dessin d'enfant is connected, it follows that any edge in E can be mapped to any other edge by the action of G . Therefore, the monodromy group G acts transitively on E .

Remark 2.7. Throughout this paper, permutations act on the left, and products are composed from right to left, i.e. $(xy) \cdot e = x \cdot (y \cdot e)$ for an edge e .

Another important observation concerning the monodromy group is that, in addition to x and y encoding the cycles of the black and white vertices, respectively, the permutation $z = (xy)^{-1}$ encodes the cycles corresponding to the faces. In fact, for each face, half of the edges incident to it form a cycle of z , while the remaining edges belong to cycles of z corresponding to the neighboring faces.

It is known that if a group G generated by two elements acts transitively on a set of edges E , then there exists a dessin d'enfant whose monodromy group is isomorphic to G [Sco24, Theorem 3.6].

Therefore, studying groups that act transitively is essential for investigating the properties of dessins d'enfants and, consequently, of algebraic curves.

Definition 2.8 (Regular dessins). [JW16, 2.1.2] A dessin d'enfant is called *regular* if its monodromy group acts freely (that is, semiregularly) on the set of its edges.

This implies that the monodromy group of a regular dessin acts freely and transitively — hence regularly — on its edges.

Lemma 2.9. *Let an (abstract) group G act on $E = \{1, \dots, n\}$. If any two of the following conditions are satisfied, the remaining one is also satisfied:*

- G acts transitively on E
- G acts freely on E
- $|G| = n$

Proof. When G acts transitively on E , for any $e \in E$ and any $e' \in E$, there exists at least one element of G sending e to e' , and hence $|G| \geq n$. When G acts freely on E , for any $e, e' \in E$, there is at most one element of G sending e to e' , hence $|G| \leq n$. Therefore, if both conditions hold, we have $|G| = n$.

When G acts transitively on E and $|G| = n$, for any $e, e' \in E$ there is exactly one element of G sending e to e' , and hence G acts freely on E .

Conversely, when G acts freely on E and $|G| = n$, the images of any $e \in E$ under the elements of G are all distinct and exhaust E . Therefore, G acts transitively on E . $\square$

Although we are not aware of a reference in which the following lemma is proved explicitly, the proof is straightforward.

Lemma 2.10. *A dessin d'enfant with n edges is regular if and only if the order of its monodromy group is n .*

Proof. Let $\mathcal{D}$ be a dessin d'enfant with n edges. Denote the set of edges by E and the monodromy group by G .

If $\mathcal{D}$ is regular, then G acts transitively and freely on E , and hence $|G| = n$ by Lemma 2.9.

Conversely, if $|G| = n$, then since G acts transitively on E , it also acts freely on E by Lemma 2.9. Therefore, $\mathcal{D}$ is regular. $\square$

Definition 2.11 (Automorphism group of a dessin d'enfant). [JW16, 2.1.2] For a dessin d'enfant $\mathcal{D}$, we define its *automorphism* to be a permutation of the set E of $\mathcal{D}$ which preserves the cyclic order of edges around each vertex, that is, which commutes with x and y , or equivalently, commutes with G . Thus we can define an *automorphism group* of G as the centralizer:

$$\text{Aut } \mathcal{D} := C_{\text{Sym}(E)}(G) = \{c \in \text{Sym}(E) \mid cg = gc \text{ for all } g \in G\}$$

$$= \{c \in \text{Sym}(E) \mid cx = xc, cy = yc\}.$$

When $\text{Aut } \mathcal{D} \cong \{1\}$, it is said that $\mathcal{D}$ has a *trivial* automorphism group.

Since $\langle x, y \rangle = \langle x, z \rangle = \langle y, z \rangle$, where $z = (xy)^{-1}$, both the monodromy group and the automorphism group are invariant under any permutation of black vertices, white vertices, and faces.

Proposition 2.12. *Let $\mathcal{D}$ be a dessin d'enfant with edge set E , monodromy group G , and automorphism group $\text{Aut } \mathcal{D}$. Then the following statements hold.*

- (1) $\text{Aut } \mathcal{D}$ acts freely on E .
- (2) $|\text{Aut } \mathcal{D}|$ divides $|E|$.
- (3) $\text{Aut } \mathcal{D}$ acts transitively on E if and only if $\mathcal{D}$ is regular.

Proof.

(1) Let $c \in \text{Aut } \mathcal{D}$ and $e \in E$, and suppose that $c \cdot e = e$. Let $e' \in E$ be arbitrary.

Since G acts transitively on E , there exists an element $g \in G$ such that $e' = g \cdot e$. Then

$$c \cdot e' = c \cdot (g \cdot e) = (cg) \cdot e = (gc) \cdot e = g \cdot (c \cdot e) = g \cdot e = e'.$$

Since $e' \in E$ was arbitrary, it follows that c fixes every element of E . Hence c is the identity element in $\text{Sym}(E)$.

Consequently,

$$\text{for all } c \in \text{Aut } \mathcal{D} \text{ and } e \in E, \quad c \cdot e = e \Rightarrow c = \text{id},$$

which shows that $\text{Aut } \mathcal{D}$ acts freely on E .

(2) By (1), $\text{Aut } \mathcal{D}$ acts freely on E . Therefore, for any $e \in E$, its orbit satisfies $|\text{Aut } \mathcal{D} \cdot e| = |\text{Aut } \mathcal{D}|$.

Since E decomposes as a disjoint union of $\text{Aut } \mathcal{D}$ -orbits, there exist elements $e_1, \dots, e_k \in E$ such that

$$E = \coprod_{i=1}^k \text{Aut } \mathcal{D} \cdot e_i.$$

Taking cardinalities, we obtain $|E| = k |\text{Aut } \mathcal{D}|$. Hence $|\text{Aut } \mathcal{D}|$ divides $|E|$, as claimed.

(3) Assume that $\text{Aut } \mathcal{D}$ acts transitively on E . Let $C := C_{\text{Sym}(E)}(\text{Aut } \mathcal{D})$ be the centralizer of $\text{Aut } \mathcal{D}$ in $\text{Sym}(E)$. By replacing G , $\text{Aut } \mathcal{D}$ in (1) with $\text{Aut } \mathcal{D}$, C , respectively, we see that C acts freely on E . Since every element of G commutes with every element of $\text{Aut } \mathcal{D}$, we have $G \subset C$. Hence G acts freely on E , and therefore $\mathcal{D}$ is regular.

Conversely, assume that $\mathcal{D}$ is regular, and fix an arbitrary edge $e_0 \in E$. Since G acts freely and transitively on E , for any $e' \in E$ there exists a unique element $g \in G$ such that $g \cdot e_0 = e'$. For each $e \in E$, we define a map $\varphi_e: E \rightarrow E$ by setting

$$\varphi_e(e') := g \cdot e, \tag{2.7}$$

where $g \in G$ is the unique element satisfying $g \cdot e_0 = e'$.

Since the action of G is free, the element g is uniquely determined by e' , and hence $g \cdot e$ is uniquely determined by e' for each fixed e . Therefore φ_e is a bijection of E , and hence $\varphi_e \in \text{Sym}(E)$.

Take arbitrary elements $g' \in G$ and $e' \in E$. As above, we can write $e' = g \cdot e_0$ for some $g \in G$. Then

$$\varphi_e(g' \cdot e') = \varphi_e(g' \cdot (g \cdot e_0)) = \varphi_e((g'g) \cdot e_0) = (g'g) \cdot e.$$

On the other hand,

$$g' \cdot \varphi_e(e') = g' \cdot \varphi_e(g \cdot e_0) = g' \cdot (g \cdot e) = (g'g) \cdot e.$$

Hence, for each $e' \in E$, the action of φ_e commutes with that of g' . Since e' is arbitrary, this implies that, as elements of $\text{Sym}(E)$, $\varphi_e g' = g' \varphi_e$. Because g' was arbitrary in G , we conclude that φ_e lies in the centralizer of G , that is, $\varphi_e \in \text{Aut } \mathcal{D}$.

By definition of φ_e , we have $\varphi_e(e_0) = e$. Thus, for any $e_0, e \in E$, there exists an element $\varphi_e \in \text{Aut } \mathcal{D}$ such that $\varphi_e(e_0) = e$. Therefore, $\text{Aut } \mathcal{D}$ acts transitively on E . $\square$

The following proposition is a basic result describing the relationship between regularity and automorphism groups.

Proposition 2.13. *A dessin d'enfant $\mathcal{D}$ is regular if and only if its monodromy group G is isomorphic to $\text{Aut } \mathcal{D}$.*

Proof. Assume that $\mathcal{D}$ is regular, and let n be the number of edges. Since the monodromy group G acts freely and transitively on E , it follows from Lemma 2.9 that $|G| = n$. By Proposition 2.12 (1) and (3), the automorphism group $\text{Aut } \mathcal{D}$ also acts freely and transitively on E . Hence, again by Lemma 2.9, $|\text{Aut } \mathcal{D}| = n$. Therefore $|\text{Aut } \mathcal{D}| = |G|$.

As explained in Proposition 2.12 (3), once we fix an edge $e_0 \in E$, the maps φ_e for $e \in E$ defined in (2.7) can be regarded as elements of $\text{Aut } \mathcal{D}$. We now define a map $\rho: G \rightarrow \text{Aut } \mathcal{D}$ by

$$\rho(g) := \varphi_{g^{-1} \cdot e_0} \quad (g \in G).$$

More explicitly, for each $e' \in E$, choose $g' \in G$ such that $g' \cdot e_0 = e'$, and define

$$\varphi_{g^{-1} \cdot e_0}(e') := g' \cdot (g^{-1} \cdot e_0) = (g'g^{-1}) \cdot e_0,$$

then we set $\rho(g) = \varphi_{g^{-1} \cdot e_0}$.

This defines a map $\rho: G \rightarrow \text{Aut } \mathcal{D}$, which will be used to identify G and $\text{Aut } \mathcal{D}$ in the regular case.

Since the action of G on E is free, the element $(g'g^{-1}) \cdot e_0$ depends on g and is distinct for different g . Hence the element $\rho(g) = \varphi_{g^{-1} \cdot e_0}$ is distinct for each $g \in G$. In other words, ρ is injective. Together with $|\text{Aut } \mathcal{D}| = |G|$, this implies that ρ is bijective.

Now let $g, h \in G$ and $e' \in E$. Choose $g' \in G$ such that $g' \cdot e_0 = e'$. Then

$$\rho(g) \circ \rho(h)(e') = \varphi_{g^{-1} \cdot e_0}(\varphi_{h^{-1} \cdot e_0}(e')) = \varphi_{g^{-1} \cdot e_0}((g'h^{-1}) \cdot e_0) = (g'h^{-1}g^{-1}) \cdot e_0.$$

On the other hand,

$$\rho(gh)(e') = \varphi_{(gh)^{-1} \cdot e_0}(e') = (g'(gh)^{-1}) \cdot e_0 = (g'h^{-1}g^{-1}) \cdot e_0.$$

Comparing this with the previous computation, we obtain

$$\rho(g) \circ \rho(h) = \rho(gh) \text{ for all } g, h \in G.$$

Hence ρ is a group homomorphism. Since ρ is bijective, it follows that ρ is an isomorphism. Therefore, $\text{Aut } \mathcal{D} \cong G$.

Conversely, suppose that $\text{Aut } \mathcal{D} \cong G$. Then $|G| = |\text{Aut } \mathcal{D}|$. Let $n = |E|$.

By Proposition 2.12 (1), the group $\text{Aut } \mathcal{D}$ acts freely on E . Hence, for any $e \in E$, the size of the $\text{Aut } \mathcal{D}$ -orbit of e satisfies

$$|\text{Aut } \mathcal{D} \cdot e| = |\text{Aut } \mathcal{D}| \leq n.$$

Therefore, $|G| \leq n$.

On the other hand, the monodromy group G acts transitively on E , and thus $|G| \geq n$. Consequently, we have $|G| = n$.

By Lemma 2.9, it follows that G acts freely on E . Hence the dessin $\mathcal{D}$ is regular. $\square$

A proof can also be found in [JW16, Theorem 2.1].

Note that regularity does not imply $G = \text{Aut } \mathcal{D}$; these groups act on the edges of the dessin in different ways.

Corollary 2.14. *Let $\mathcal{D}$ be a dessin d'enfant with edge set E , number of edges $n = |E|$, monodromy group G , and automorphism group $\text{Aut } \mathcal{D}$. Then the following conditions are equivalent:*

- (1) $\mathcal{D}$ is regular.
- (2) G acts freely on E .
- (3) $\text{Aut } \mathcal{D}$ acts transitively on E .
- (4) $|G| = n$.
- (5) $|\text{Aut } \mathcal{D}| = n$.
- (6) $G \cong \text{Aut } \mathcal{D}$.

Proof.

- (1) $\Leftrightarrow$ (2): This follows directly from Definition 2.8.
- (1) $\Leftrightarrow$ (3): This is a consequence of Proposition 2.12 (3).
- (2) $\Leftrightarrow$ (4): Since the monodromy group G acts transitively on E , the claim follows from Lemma 2.9.
- (3) $\Leftrightarrow$ (5): This follows from Proposition 2.12 (1) together with Lemma 2.9.
- (1) $\Leftrightarrow$ (6): This is a consequence of Proposition 2.13. □

Proposition 2.15. *If a dessin d'enfant is regular, then it has a uniform passport.*

Proof. A proof can be found in [GGD12, Proposition 4.42]. For the reader's convenience, we include another proof.

We prove the contrapositive. Suppose that the dessin $\mathcal{D}$ is not uniform. Then $\mathcal{D}$ has either two black vertices, two white vertices, or two faces whose valencies are different.

Since regularity is symmetric with respect to black vertices, white vertices, and faces, we may assume without loss of generality that these are black vertices. Let these vertices be A and B , and let their valencies be l and m , with $l < m$.

Under x^l , all the edges around A are mapped to themselves, whereas the edges around B are mapped to distinct edges. Hence $x^l \neq \text{id}$, and there exist at least two distinct elements of G that fix the edges around A . Therefore, G does not act freely on E , and $\mathcal{D}$ is not regular. □

The converse of this proposition does not hold in general. This shows that regularity exhibits a higher degree of symmetry than uniformity. This fact is one of the main topics of this paper.

Example 2.16. The uniform passport $[4^2, 2^4, 4^2]$ (genus 1) corresponds to the two dessins shown in Section 1.2, Figure 2. The dessin on the left is regular, and both its monodromy group and its automorphism group are isomorphic to $C_4 \times C_2$, the direct product of cyclic groups of orders 4 and 2, respectively. In contrast, the dessin on the right is not regular; its monodromy group is isomorphic to $(C_4 \times C_2) \rtimes C_2$ and has order 16, whereas its automorphism group has order 4 and is isomorphic to $C_2 \times C_2$.

We also recall the notions of primitive and imprimitive groups, which will play a role in investigating when a dessin has a trivial automorphism group.

Definition 2.17 (Block in a group action). [DM96, 1.5] Let G be a group acting transitively on a set E . A nonempty subset S of E is called a *block* for G if for each $g \in G$ either $g \cdot S = S$ or $(g \cdot S) \cap S = \emptyset$.

Every group acting transitively on E has E and the singletons $\{e\}$ ($e \in E$) as blocks. these are called *trivial* blocks. Any other block is called *nontrivial*.

In other words, a block is a subset that is always moved as a whole under the action of G .

Definition 2.18 (Primitive and imprimitive groups). [DM96, 1.5] Let G be a group which acts transitively on a set E . G is called *primitive* if it has no nontrivial blocks on E ; otherwise G is called *imprimitive*.

3. Estimates for the Symmetric Group

In order to investigate the behavior of automorphism groups in genus at least 2 (see Section 7), we need to obtain estimates for the number of permutations in the symmetric group S_n . In this section, we prove two theorems (Theorems 3.3 and 3.5) that provide lower and upper bounds for quantities related to monodromy groups.

3.1. Elements of Cycle Type (b^q)

Let $x = (1 \ 2 \ \dots \ n) \in S_n$ and assume that $n = bq$, where $b, q \in \mathbb{Z}_{>0}$. We define the following subsets of S_n :

- $T(b, q)$: the set of permutations in S_n with cycle type (b^q) , that is, permutations consisting of q disjoint cycles of length b ;
- $N(b, q)$: the set of elements $y \in T(b, q)$ such that xy has cycle type (n) ;
- $I(b, q)$: the set of elements $y \in T(b, q)$ such that $G = \langle x, y \rangle$ is imprimitive.

Note that the condition $y \in T(b, q)$ with xy of cycle type (n) is equivalent to requiring that $z = (xy)^{-1}$ has cycle type (n) , since xy and $(xy)^{-1}$ have the same cycle type.

In addition, for a divisor m of n with $2 \leq m < n$, let $I_m(b, q)$ denote the subset of $I(b, q)$ consisting of those elements y for which $G = \langle x, y \rangle$ is imprimitive with respect to the partition of $\{1, \dots, n\}$ into residue classes modulo m ; that is, the residue classes modulo m form a system of blocks for G .

Hereafter, we use the same notation to denote the number of elements in a set (for example, the number of elements of $T(b, q)$ is denoted by the function $T(b, q)$).

Proposition 3.1. *The number of elements in S_n of cycle type (b^q) ($n = bq$) is given by*

$$T(b, q) = \frac{n!}{b^q q!}.$$

Proof. If we write an element of cycle type (b^q) as

$$(a_1 \dots a_b)(a_{b+1} \dots a_{2b}) \cdots (a_{(q-1)b+1} \dots a_{qb}),$$

then there are $n!$ possible arrangements of these numbers. Among them, the following duplications occur as permutations:

- the b^q arrangements of elements within the q cycles of length b ,
- the $q!$ arrangements of the q cycles themselves.

Hence, we obtain $T(b, q) = n!/(b^q q!)$. □

3.2. Lower Bound for N/T

A useful tool for computing the values of $N(b, q)/T(b, q)$ is the following theorem from [GS98].

Theorem 3.2. *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_m)$ be partitions of n . Define the genus associated with the pair (λ, μ) by*

$$g = \frac{n - (l + m) + 1}{2},$$

and assume that $g \in \mathbb{Z}_{\geq 0}$.

Let $c_{\lambda\mu}^n$ denote the number of solutions $(\sigma, \rho) \in C_\lambda \times C_\mu$ to the equation $\sigma\rho = \pi$, where π is a fixed n -cycle in S_n . Then $c_{\lambda\mu}^n$ is given by

$$c_{\lambda\mu}^n = \frac{n}{z_\lambda z_\mu 2^{2g}} \sum_{\substack{g_1, g_2 \geq 0 \\ g_1 + g_2 = g}} (l + 2g_1 - 1)!(m + 2g_2 - 1)! \sum_{\substack{(i_1, \dots, i_l) \models g_1 \\ (j_1, \dots, j_m) \models g_2}} \prod_{k=1}^l \binom{\lambda_k}{2i_k + 1} \prod_{k=1}^m \binom{\mu_k}{2j_k + 1}.$$

Here $p \models n$ denotes a composition of n , that is, a finite sequence of non-negative integers summing to n , where the order of the terms matters. We adopt the convention that $\binom{a}{b} = 0$ if $b > a$.

Moreover, for a partition $\lambda = 1^{\alpha_1} \dots n^{\alpha_n}$, where α_i denotes the multiplicity of i , we define $z_\lambda = \prod_i \alpha_i! i^{\alpha_i}$.

Proof. See [GS98, Theorem 2.1]. □

Using this theorem, we establish a lower bound for $N(b, q)/T(b, q)$.

Theorem 3.3. *Let $n \geq 3$ be an integer such that $n = bq$, with $b \geq 2$, and assume that $n \equiv q \pmod{2}$. Fix $x = (1 \ 2 \ \dots \ n) \in S_n$. Then*

$$\frac{N(b, q)}{T(b, q)} \geq \frac{2}{n + 2}.$$

In particular, the inequality becomes an equality when $b = 2$.

Proof. In Theorem 3.2, for the passport $[n, b^q, n]$, we have $\lambda = (n^1)$, $l = 1$, $\mu = (b^q)$, and $m = q$. Moreover, the genus is given by $g = (n - q)/2 = q(b - 1)/2$. Hence,

$$N(b, q) = c_{(n)(b^q)}^{(n)} = \frac{1}{b^q q! 2^{2g}} \sum_{\substack{g_1, g_2 \geq 0 \\ g_1 + g_2 = g}} (2g_1)!(2g_2 + q - 1)! \binom{n}{2g_1 + 1} \sum_{(j_1, \dots, j_q) \models g_2} \prod_{k=1}^q \binom{b}{2j_k + 1}.$$

Since $T(b, q) = n!/(b^q q!)$ by Proposition 3.1, we obtain

$$\frac{N}{T} = \frac{1}{2^{2g} n!} \sum_{\substack{g_1, g_2 \geq 0 \\ g_1 + g_2 = g}} (2g_1)!(2g_2 + q - 1)! \binom{n}{2g_1 + 1} \sum_{(j_1, \dots, j_q) \models g_2} \prod_{k=1}^q \binom{b}{2j_k + 1}.$$

For the terms involving g_1 and g_2 in the outer summation, we have

$$\frac{1}{n!} (2g_1)!(2g_2 + q - 1)! \binom{n}{2g_1 + 1} = \frac{1}{n!} (2g_1)!(2g_2 + q - 1)! \frac{n!}{(2g_1 + 1)!(n - 2g_1 - 1)!}$$

$$= \frac{(2g_2 + q - 1)!}{(2g_1 + 1)(n - 2g_1 - 1)!}.$$

Since $g_1 + g_2 = g = (n - q)/2$, we have $q = n - 2g_1 - 2g_2$. Hence,

$$\begin{aligned} \frac{(2g_2 + q - 1)!}{(2g_1 + 1)(n - 2g_1 - 1)!} &= \frac{(2g_2 + (n - 2g_1 - 2g_2) - 1)!}{(2g_1 + 1)(n - 2g_1 - 1)!} \\ &= \frac{1}{2g_1 + 1} = \frac{1}{2(g - g_2) + 1}. \end{aligned}$$

Therefore,

$$\frac{N}{T} = \frac{1}{2^{2g}} \sum_{g_2=0}^g \frac{1}{2(g - g_2) + 1} \sum_{(j_1, \dots, j_q) \models g_2} \prod_{k=1}^q \binom{b}{2j_k + 1} \quad \left(g = \frac{n - q}{2}\right).$$

Let

$$A_{g_2} := \sum_{(j_1, \dots, j_q) \models g_2} \prod_{k=1}^q \binom{b}{2j_k + 1},$$

then

$$\frac{N}{T} = \frac{1}{2^{2g}} \sum_{g_2=0}^g \frac{A_{g_2}}{2(g - g_2) + 1}.$$

Consider the polynomial

$$P_b(x) := \sum_{j=0}^{\lfloor (b-1)/2 \rfloor} \binom{b}{2j+1} x^j.$$

Then we may write

$$P_b(x)^q = \sum_{g_2=0}^{q\lfloor (b-1)/2 \rfloor} A_{g_2} x^{g_2}, \quad A_{g_2} = [x^{g_2}] P_b(x)^q,$$

where $[x^{g_2}] P_b(x)^q$ denotes the coefficient of x^{g_2} in $P_b(x)^q$.

Moreover,

$$\begin{aligned} P_b(1) &= \sum_{j=0}^{\lfloor (b-1)/2 \rfloor} \binom{b}{2j+1} = \sum_{\substack{l=1 \\ l: \text{ odd}}}^b \binom{b}{l} = \frac{1}{2} \left(\sum_{l=0}^b \binom{b}{l} - \sum_{l=0}^b (-1)^l \binom{b}{l} \right) \\ &= \frac{1}{2} ((1+1)^b - (1-1)^b) = 2^{b-1}. \end{aligned} \tag{3.1}$$

Therefore,

$$\sum_{g_2=0}^{q\lfloor (b-1)/2 \rfloor} A_{g_2} = P_b(1)^q = 2^{q(b-1)} = 2^{2g}.$$

If we define $\pi(g_2) := A_{g_2}/2^{2g}$, then

$$\sum_{g_2=0}^{q\lfloor (b-1)/2 \rfloor} \pi(g_2) = 1.$$

Hence, $\pi(g_2)$ can be regarded as a probability mass function.

Since

$$\frac{N}{T} = \frac{1}{2^{2g}} \sum_{g_2=0}^g \frac{A_{g_2}}{2(g-g_2)+1} = \sum_{g_2=0}^g \frac{\pi(g_2)}{2(g-g_2)+1},$$

if a random variable X follows the probability distribution $\mathbb{P}(X = g_2) = \pi(g_2)$, then N/T can be expressed in terms of its expected value as

$$\frac{N}{T} = \mathbb{E} \left[\frac{1}{2(g-X)+1} \right]. \quad (3.2)$$

On the other hand, since

$$\sum_{j=0}^{\lfloor (b-1)/2 \rfloor} \binom{b}{2j+1} = 2^{b-1},$$

we can define a random variable J by

$$\mathbb{P}(J = j) = \begin{cases} \frac{\binom{b}{2j+1}}{2^{b-1}} & (0 \leq j \leq \lfloor (b-1)/2 \rfloor) \\ 0 & (\text{otherwise}) \end{cases}. \quad (3.3)$$

Let $J_1, \dots, J_q$ be independent random variables following this distribution, and define $S_q := J_1 + \dots + J_q$. Then

$$\mathbb{P}(S_q = g_2) = \sum_{j_1+\dots+j_q=g_2} \prod_{k=1}^q \mathbb{P}(J_k = j_k) = \frac{1}{2^{q(b-1)}} \sum_{j_1+\dots+j_q=g_2} \prod_{k=1}^q \binom{b}{2j_k+1} = \frac{A_{g_2}}{2^{2g}} = \pi(g_2).$$

Therefore, S_q follows the same probability distribution as in (3.2). Then we have

$$\frac{N}{T} = \mathbb{E} \left[\frac{1}{2(g-S_q)+1} \right]. \quad (3.4)$$

Here,

$$\begin{aligned} \mathbb{E}[J] &= \sum_{j \geq 0} j \cdot \mathbb{P}(J = j) = \frac{1}{2^{b-1}} \sum_{j=0}^{\lfloor (b-1)/2 \rfloor} \binom{b}{2j+1} \\ &= \frac{1}{2^{b-1}} \sum_{\substack{k=1 \\ k: \text{ odd}}} \frac{k-1}{2} \binom{b}{k} \quad (k = 2j+1) \\ &= \frac{1}{2^b} \sum_{\substack{k=1 \\ k: \text{ odd}}}^b k \binom{b}{k} - \frac{1}{2^b} \sum_{\substack{k=1 \\ k: \text{ odd}}}^b \binom{b}{k}. \end{aligned} \quad (3.5)$$

From (3.1),

$$\sum_{\substack{k=1 \\ k: \text{ odd}}}^b \binom{b}{k} = 2^{b-1}. \quad (3.6)$$

Moreover, starting from

$$(1+x)^b = \sum_{k=0}^b \binom{b}{k} x^k,$$

differentiating both sides with respect to x and then multiplying by x gives

$$bx(1+x)^{b-1} = \sum_{k=0}^b k \binom{b}{k} x^k = \sum_{k=1}^b k \binom{b}{k} x^k.$$

Substituting $x = 1$ and $x = -1$ respectively, we obtain

$$\begin{aligned} 2^{b-1}b &= \sum_{k=1}^b k \binom{b}{k} = \sum_{\substack{k=1 \\ k: \text{ odd}}}^b k \binom{b}{k} + \sum_{\substack{k=1 \\ k: \text{ even}}}^b k \binom{b}{k}, \\ 0 &= \sum_{k=1}^b k \binom{b}{k} (-1)^k = - \sum_{\substack{k=1 \\ k: \text{ odd}}}^b k \binom{b}{k} + \sum_{\substack{k=1 \\ k: \text{ even}}}^b k \binom{b}{k}. \end{aligned}$$

Subtracting these two equations and dividing by 2, we obtain

$$\sum_{\substack{k=1 \\ k: \text{ odd}}}^b k \binom{b}{k} = \frac{2^{b-1}b - 0}{2} = 2^{b-2}b. \quad (3.7)$$

From (3.5), (3.6) and (3.7), we obtain

$$\mathbb{E}[J] = \frac{2^{b-2}b}{2^b} - \frac{2^{b-1}}{2^b} = \frac{b-2}{4}. \quad (3.8)$$

Hence,

$$g - \mathbb{E}[S_q] = g - q\mathbb{E}[J] = g - \frac{q(b-2)}{4} = \frac{q(b-1)}{2} - \frac{q(b-2)}{4} = \frac{bq}{4} = \frac{n}{4}.$$

For (3.4), consider the function $f(x) = 1/(2(g-x)+1)$ ($0 \leq x \leq g$). Then,

$$f''(x) = \frac{8}{(2(g-x)+1)^3} > 0.$$

Hence, $f(x)$ is a convex function. By Jensen's inequality, we have

$$\begin{aligned} \frac{N}{T} &= \mathbb{E}[f(S_q)] \geq f(\mathbb{E}[S_q]) = \frac{1}{2(g - \mathbb{E}[S_q]) + 1} \\ &= \frac{1}{2 \cdot \frac{n}{4} + 1} = \frac{2}{n+2}. \end{aligned}$$

In particular, when $b = 2$, it follows from (3.8) that $\mathbb{E}[J] = 0$. As the random variable J is always non-negative by (3.3), in this case we conclude that $J \equiv 0$.

Consequently, $S_q = J_1 + \dots + J_q \equiv 0$. Therefore, by (3.4),

$$\frac{N}{T} = \mathbb{E}\left[\frac{1}{2(g - S_q) + 1}\right] = \frac{1}{2g+1} = \frac{1}{2 \cdot \frac{q(b-1)}{2} + 1} = \frac{1}{q+1} = \frac{2}{2q+2} = \frac{2}{n+2}.$$

Therefore, the inequality is sharp, and equality holds precisely when $b = 2$. $\square$

3.3. Computation of I_m

In order to find an upper bound for $I(b, q)/T(b, q)$, we divide the problem according to the divisors m of n with $2 \leq m < n$.

Lemma 3.4. *Let $G \leq S_n$ be a subgroup containing $x = (1\ 2\ \dots\ n)$. If the natural action of G on $E = \{1, \dots, n\}$ is imprimitive, then every nontrivial block of G is of the form*

$$B_{m,j} = \left\{ j, j+m, j+2m, \dots, j + \left(\frac{n}{m} - 1 \right) m \right\}$$

for some divisor m of n with $2 \leq m < n$ and some $1 \leq j \leq m$, where the elements are taken modulo n .

Moreover, the cyclic subgroup $H = \langle x \rangle$ has all of the above sets $B_{m,j}$ as its nontrivial blocks.

Proof. In the following, the elements of E are taken modulo n .

Let $B \subset E$ be a nontrivial block of G . Since B has at least two elements by Definition 2.17, we may choose $j, j+d \in B$ with $j \in E$ and $0 < d < n$. Then $x^d \in G$ and $x^d \cdot j = j+d \in B$. Hence, since $(x^d \cdot B) \cap B \neq \emptyset$, it follows from Definition 2.17 that $x^d \cdot B = B$.

Applying x^d once more, we obtain $x^d \cdot (j+d) = j+2d \in B$. Repeating this argument, it follows that

$$\{j + kd \mid k \in \mathbb{Z}_{\geq 0}\} \subset B.$$

Let $m_0 = \gcd(n, d)$. Working modulo n , we have

$$\begin{aligned} \{j + kd \mid k \in \mathbb{Z}_{\geq 0}\} &= \{j + km_0 \mid k \in \mathbb{Z}_{\geq 0}\} \\ &= \left\{ j, j+m_0, j+2m_0, \dots, j + \left(\frac{n}{m_0} - 1 \right) m_0 \right\} \\ &= B_{m_0,j}. \end{aligned}$$

Hence,

$$B_{m_0,j} \subset B.$$

If $B_{m_0,j} \subsetneq B$, we may choose $j+d' \in B \setminus B_{m_0,j}$ with $m_0 \nmid d'$. Applying the above argument again, we obtain

$$\{j + kd' \mid k \in \mathbb{Z}_{\geq 0}\} \subset B,$$

and hence

$$B_{m_0,j} \cup \{j + kd' \mid k \in \mathbb{Z}_{\geq 0}\} \subset B.$$

Let $m_1 = \gcd(m_0, d')$. Then $m_1 \mid n$ and we obtain

$$\begin{aligned} B_{m_0,j} \cup \{j + kd' \mid k \in \mathbb{Z}_{\geq 0}\} &= \{j + km_0 + k'd' \mid k, k' \in \mathbb{Z}_{\geq 0}\} \\ &= \{j + km_1 \mid k \in \mathbb{Z}_{\geq 0}\} = B_{m_1,j}. \end{aligned}$$

Hence,

$$B_{m_0,j} \subsetneq B_{m_1,j} \subset B.$$

This procedure can be repeated for m_i ($i \geq 0$), and since $B \subset E$ is finite, the process terminates. Consequently, we eventually obtain

$$B_{m,j} = B$$

for some divisor m of n with $m < n$.

If $m = 1$, then $B = E$, which is not a nontrivial block. Therefore, we must have $2 \leq m < n$.

Since $x \in H$, the nontrivial blocks of H are also restricted to those of the form $B_{m,j}$. Conversely, let m and j satisfy $m \mid n$, $2 \leq m < n$, and $1 \leq j \leq m$. For any $x^d \in H$, we have

$$x^d \cdot B_{m,j} = \left\{ j + d, j + m + d, j + 2m + d, \dots, j + \left(\frac{n}{m} - 1 \right) m + d \right\} = B_{m,j+d}.$$

If $m \mid d$, then $B_{m,j+d} = B_{m,j}$, since for each $0 \leq k \leq n/m - 1$, there exists $0 \leq k' \leq n/m - 1$ such that $j + km + d = j + k'm$. Hence $x^d \cdot B_{m,j} = B_{m,j}$.

If $m \nmid d$, then for each $0 \leq k \leq n/m - 1$, we have $j + km + d \neq j + k'm$ for any $0 \leq k' \leq n/m - 1$. Hence $(x^d \cdot B_{m,j}) \cap B_{m,j} = \emptyset$.

Therefore, each $B_{m,j}$ satisfies the defining condition of a block of H . Since $1 < |B_{m,j}| = n/m < n$, every $B_{m,j}$ is a nontrivial block of H . $\square$

By the above lemma, $G = \langle x, y \rangle$ is imprimitive if and only if $\langle y \rangle$ admits the residue classes modulo some $m \mid n$ ($2 \leq m < n$) as a block system. Equivalently, $y \in I_m$ (see Section 3.1).

We now consider what kinds of such y can occur, taking the case $(n, b, q, m) = (18, 3, 6, 6)$ as an example. In this case, the blocks are the residue classes of $\{1, \dots, 18\}$ modulo 6. Let these blocks be $B_0, \dots, B_5$ (where $B_0 = \{6, 12, 18\}$). The permutations y that preserve these blocks fall into the following three types:

- (1) $(B_1 B_1 B_1)(B_2 B_2 B_2)(B_3 B_3 B_3)(B_4 B_4 B_4)(B_5 B_5 B_5)(B_0 B_0 B_0)$
e.g. $(1\ 7\ 13)(2\ 14\ 8)(3\ 9\ 15)(4\ 10\ 16)(5\ 17\ 11)(6\ 12\ 18)$
- (2) $(B_1 B_2 B_3)(B_1 B_2 B_3)(B_1 B_2 B_3)(B_4 B_5 B_0)(B_4 B_5 B_0)(B_4 B_5 B_0)$
e.g. $(1\ 2\ 15)(7\ 8\ 9)(13\ 14\ 3)(4\ 5\ 6)(10\ 17\ 12)(16\ 11\ 18)$
and those obtained by permuting the blocks.
- (3) $(B_1 B_1 B_1)(B_2 B_2 B_2)(B_3 B_3 B_3)(B_4 B_5 B_0)(B_4 B_5 B_0)(B_4 B_5 B_0)$
e.g. $(1\ 7\ 13)(2\ 8\ 14)(3\ 15\ 9)(10\ 5\ 6)(4\ 17\ 12)(16\ 11\ 18)$
and those obtained by permuting the blocks.

In case (1), all permutations are performed within the same blocks. In case (2), the six blocks are divided into two groups of three, and permutations are performed within each group according to a fixed order. Case (3) represents a mixed type of the previous two.

These correspond to the partitions of $m = 6$: (1) $1 \times 6 = 6$, (2) $3 \times 2 = 6$, and (3) $1 \times 3 + 3 \times 1 = 6$. If there are t_i cycles, each involving d_i distinct blocks, then $\sum_i d_i t_i = m$. For the permutation to cycle within each group of d_i blocks, it must hold that $d_i \mid b$. Furthermore, since each block (of which there are $n/m = bq/m$) must rotate its elements in units of b/d_i , it must also hold $b/d_i \mid bq/m$, i.e. $m \mid d_i q$. The partitions of 6 satisfying these conditions are $\{(d_i, t_i)\} = \{(1, 6)\}, \{(3, 2)\}, \{(1, 3), (3, 1)\}$, corresponding precisely to the three types described above.

Theorem 3.5. *Fix $x = (1\ 2\ \dots\ n) \in S_n$. Among the elements $y \in S_n$ of cycle type (b^q) , the number of those for which $G = \langle x, y \rangle$ becomes an imprimitive group with respect to residue classes modulo m ($m \mid n$, $2 \leq m < n$) is given by*

$$I_m(b, q) = \frac{m!}{b^q} \left(\left(\frac{n}{m} \right)! \right)^m \sum_{\{(d_i, t_i)\}} \prod_i \frac{1}{d_i^{t_i} t_i!} \left(\frac{d_i^{d_i q}}{\left(\frac{d_i q}{m} \right)!} \right)^{t_i},$$

where the summation on the right-hand side is taken over all partitions $\{(d_i, t_i)\}$ of m satisfying

$$\sum_i d_i t_i = m, \quad d_i \text{ are all distinct}, \quad d_i \mid b, \quad m \mid d_i q \text{ for all } i. \quad (3.9)$$

Proof. By Lemma 3.4, it suffices to count the number of elements y of cycle type (b^q) whose blocks under the action of $\langle y \rangle$ are exactly the residue classes modulo m . As discussed above, such a permutation y determines a partition of m , as defined by (3.9).

For each i , there are t_i cycles of length b , each containing d_i distinct blocks. For these d_i blocks to circulate within a cycle, we must have $d_i \mid b$. Moreover, since each block (and there are $n/m = bq/m$ such blocks) rotates its elements in units of b/d_i , we must have $b/d_i \mid bq/m$, that is, $m \mid d_i q$.

Fix a partition $\{(d_i, t_i)\}$ of m . First, determine the assignment of blocks corresponding to $d_1 \cdot t_1, d_2 \cdot t_2, \dots$ and the order of each set of d_i blocks (this corresponds to the block arrangement in the preceding example). There are $m!$ ways to arrange the m blocks in total. For each pair (d_i, t_i) , there are d_i ways to represent the same permutation of d_i distinct blocks, and $t_i!$ ways to order the t_i types. Hence, the number of possible assignments and orderings of blocks is given by

$$\frac{m!}{\prod_i d_i^{t_i} t_i!}. \quad (3.10)$$

In a cycle of length b containing d_i distinct blocks, each cycle consists of d_i blocks, with b/d_i elements from each block. The same type of cycle appears $(bq/m)/(b/d_i) = d_i q/m$ times. Focusing on one of these d_i blocks, since it contains $n/m = bq/m$ elements and these are arranged into $d_i q/m$ cycles, the number of such configurations is, by Proposition 3.1,

$$T\left(\frac{\frac{bq}{m}}{\frac{d_i q}{m}}, \frac{d_i q}{m}\right) = \frac{\left(\frac{n}{m}\right)!}{\left(\frac{b}{d_i}\right)^{\frac{d_i q}{m}} \left(\frac{d_i q}{m}\right)!}.$$

The number of ways to place the elements of the remaining $d_i - 1$ blocks after those of the first block is $((n/m)!)^{d_i-1}$, since each block contains n/m elements. Hence, the total number of ways to arrange the elements of these d_i blocks is

$$\frac{\left(\left(\frac{n}{m}\right)!\right)^{d_i}}{\left(\frac{b}{d_i}\right)^{\frac{d_i q}{m}} \left(\frac{d_i q}{m}\right)!}.$$

There are t_i such types of blocks, and the number of possible assignments and orderings of blocks is given by (3.10). Summing over all partitions $\{(d_i, t_i)\}$ of m , we obtain

$$I_m(b, q) = \sum_{\{(d_i, t_i)\}} \prod_i \frac{m!}{d_i^{t_i} t_i!} \left(\frac{\left(\left(\frac{n}{m}\right)!\right)^{d_i}}{\left(\frac{b}{d_i}\right)^{\frac{d_i q}{m}} \left(\frac{d_i q}{m}\right)!} \right)^{t_i} \quad (n = bq),$$

where the summation is taken over all partitions $\{(d_i, t_i)\}$ of m satisfying (3.9).

The total products of the terms $(n/m)!$ and b are

$$\begin{aligned} \prod_i \left(\left(\frac{n}{m}\right)!\right)^{d_i t_i} &= \left(\left(\frac{n}{m}\right)!\right)^{\sum_i d_i t_i} = \left(\left(\frac{n}{m}\right)!\right)^m, \\ \prod_i \left(\frac{1}{b^{\frac{d_i q}{m}}}\right)^{t_i} &= \left(\frac{1}{b}\right)^{\frac{q}{m} \sum_i d_i t_i} = \frac{1}{b^q}. \end{aligned}$$

Also, $m!$ can be factored out. Then we obtain

$$I_m(b, q) = \frac{m!}{b^q} \left(\left(\frac{n}{m}\right)!\right)^m \sum_{\{(d_i, t_i)\}} \prod_i \frac{1}{d_i^{t_i} t_i!} \left(\frac{d_i^{\frac{q}{m}}}{\left(\frac{d_i q}{m}\right)!}\right)^{t_i}.$$

□

4. Genus 0

From this section onward, we study the regularity and automorphism groups of uniform dessins d'enfants by genus.

In the case of uniform dessins d'enfants of genus 0, the situation is simple.

Proposition 4.1. *Every uniform dessin d'enfant of genus 0 is regular, and its automorphism group is isomorphic to its monodromy group and has order n (the number of edges).*

Proof. Let a uniform dessin d'enfant of genus 0 have the passport $[a^p, b^q, c^r]$ with $n = pa = qb = rc$. Since regularity is invariant under permutations of black vertices, white vertices, and faces, we may assume $c \geq a \geq b$ (and hence $r \leq p \leq q$).

Since the genus is 0, by (2.3) we have

$$p + q + r = \frac{n}{a} + \frac{n}{b} + \frac{n}{c} = n + 2,$$

and hence

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1 + \frac{2}{n} > 1.$$

Since $c \geq a \geq b$, it follows that $b < 3$, and therefore $b = 1$ or $b = 2$.

When $b = 1$, we have

$$\frac{1}{a} + \frac{1}{c} = \frac{2}{n},$$

which implies

$$\frac{p}{n} + \frac{r}{n} = \frac{2}{n},$$

and hence

$$p = r = 1.$$

Therefore, the passport is $[n, 1^n, n]$.

When $b = 2$, the integer n is even, and we have

$$\frac{1}{a} + \frac{1}{c} = \frac{1}{2} + \frac{2}{n} > \frac{1}{2}.$$

Since $2 \leq a \leq c$, it follows that $a = 2$ or $a = 3$.

- When $a = 2$, we have $c = n/2$, and hence the passport is $[2^m, 2^m, m^2]$ (with $n = 2m$).
- When $a = 3$, we have $c = 6n/(n + 12)$. The pairs (n, c) satisfying this equation are $(12, 3)$, $(24, 4)$, and $(60, 5)$.

From the above, all uniform dessins d'enfants of genus 0 are included in the following:

$$[n, 1^n, n], [2^m, 2^m, m^2] (n = 2m), [3^4, 2^6, 3^4], [3^8, 2^{12}, 4^6], [3^{20}, 2^{30}, 5^{12}]. \quad (4.1)$$

The dessins corresponding to these passports are shown in Figure 7. They correspond to the shapes of a star, an n -gon, a tetrahedron, a hexahedron, and an icosahedron, respectively.

It is easy to show that no other dessins exist for the passports listed in (4.1), and that their monodromy groups and automorphism groups are isomorphic. These groups are C_n (the cyclic group of order n), D_m (the dihedral group of order $2m$), A_4 (the alternating

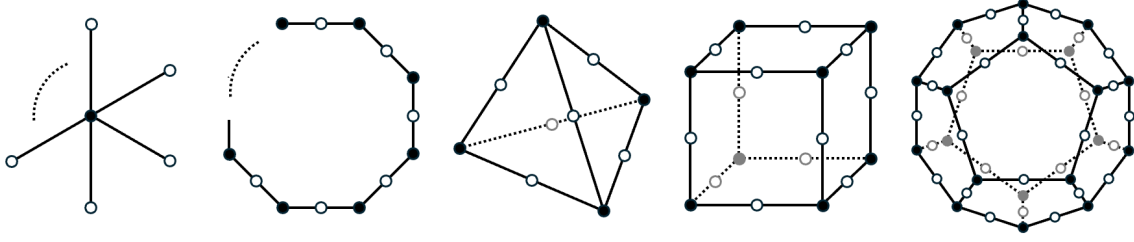


FIGURE 7. Uniform dessins d'enfants of genus 0

group of degree 4), S_4 (the symmetric group of degree 4), and A_5 (the alternating group of degree 5), respectively². Therefore, all uniform dessins d'enfants of genus 0 are regular.

Moreover, by Proposition 2.13 and Lemma 2.10, it follows that the automorphism group of every uniform dessin d'enfant of genus 0 is isomorphic to its monodromy group and has order n . $\square$

Thus, when restricted to genus 0, regularity and uniformity are equivalent.

5. Genus 1

An explicit analysis of regularity and automorphism groups of genus-1 dessins indicates that, except for passports containing a partition of the form n^1 , all such passports appear to admit non-regular dessins, while trivial automorphism groups appear not to arise in genus 1. In this section, we prove these statements.

In general, for a dessin d'enfant $\mathcal{D}$ whose valencies of the black vertices, the white vertices, and the faces have least common multiples a , b , and c , respectively, we consider the triangle group

$$\Delta = \Delta(a, b, c) = \langle X, Y, Z \mid X^a = Y^b = Z^c = XYZ = 1 \rangle.$$

Then, for the monodromy group $G = \langle x, y \rangle$ of $\mathcal{D}$ defined in Definition 2.6, one can define an action of Δ on $\mathcal{D}$ via the group homomorphism

$$\varphi: X \mapsto x, Y \mapsto y, Z \mapsto (xy)^{-1},$$

so that G may be viewed as a quotient of Δ .

Let $\Delta_e := \ker \varphi$. Then Δ_e is the subgroup of Δ consisting of all elements that fix every edge of $\mathcal{D}$, and it satisfies

$$[\Delta : \Delta_e] < \infty.$$

Moreover,

$$\mathcal{D} \text{ is regular} \Leftrightarrow \Delta_e \triangleleft \Delta. \quad (5.1)$$

See [JW16, Theorem 3.13].

For a dessin d'enfant $\mathcal{D}$ with a uniform passport $[a^p, b^q, c^r]$ with $n = pa = qb = rc$, the corresponding triangle group is $\Delta(a, b, c)$. If $\mathcal{D}$ has genus 1, then by (2.3) we have

$$p + q + r = \frac{n}{a} + \frac{n}{b} + \frac{n}{c} = n,$$

and hence

² By exchanging black vertices and faces, the passport $[4^6, 2^{12}, 3^8]$ corresponds to an octahedron and $[5^{12}, 2^{30}, 3^{20}]$ to a dodecahedron; their monodromy (and hence automorphism) groups are S_4 and A_5 , respectively.

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

This case is referred to as the Euclidean case of triangle groups. In this situation, Δ is infinite, and Δ_e is torsion-free (that is, it contains no nontrivial element of finite order).

Assuming $a \leq b \leq c$, the only possible triples are

$$(a, b, c) = (2, 4, 4), (3, 3, 3), (2, 3, 6).$$

Hence the uniform passports of genus 1 are exactly

$$[2^{2m}, 4^m, 4^m], [3^m, 3^m, 3^m], [2^{3m}, 3^{2m}, 6^m] \quad (m \in \mathbb{Z}_{>0}),$$

together with the ones obtained from these by permuting the three parts.

For these passports, if we set

$$d := \max\{a, b, c\},$$

then $d = 4, 3, 6$, respectively, and the number of edges of the dessin (equivalently, the degree of the associated Belyi function) is always md .

Consider a triangle ABC in the Euclidean plane with angles $\angle A = \pi/a$, $\angle B = \pi/b$, and $\angle C = \pi/c$. The generators X, Y, Z of the triangle group Δ correspond to the following rotations of the whole plane:

$$\begin{aligned} X &: \text{rotation by } \frac{2\pi}{a} \text{ about the point } A, \\ Y &: \text{rotation by } \frac{2\pi}{b} \text{ about the point } B, \\ Z &: \text{rotation by } \frac{2\pi}{c} \text{ about the point } C. \end{aligned}$$

Moreover, Δ contains as a normal subgroup the group $T \cong \mathbb{Z}^2$ consisting of all translations of the plane.

We fix the vertices as $A = (0, 0)$ and $B = (1, 0)$, and choose the remaining vertex C , the translation subgroup T , and the quotient Δ/T as follows:

- $\Delta = \Delta(2, 4, 4)$:

$$C: (0, 1), \quad T: (u, v) \mapsto (u + 2k, v + 2l) \quad (k, l \in \mathbb{Z}), \quad \Delta/T \cong C_4 \quad (5.2)$$

- $\Delta = \Delta(3, 3, 3)$:

$$\begin{aligned} C: \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad T: (u, v) \mapsto \left(u + \frac{3}{2}(k + l), v + \frac{\sqrt{3}}{2}(k - l)\right) \quad (k, l \in \mathbb{Z}), \\ \Delta/T \cong C_3 \end{aligned} \quad (5.3)$$

- $\Delta = \Delta(2, 3, 6)$:

$$\begin{aligned} C: (0, \sqrt{3}), \quad T: (u, v) \mapsto (u + 3(k + l), v + \sqrt{3}(k - l)) \quad (k, l \in \mathbb{Z}), \\ \Delta/T \cong C_6 \end{aligned} \quad (5.4)$$

In each case, the elements of Δ fall into the following categories:

- (1) the identity transformation;
- (2) rotations about a point;
- (3) translations (elements of $T \setminus \{\text{id}\}$);
- (4) rotations about a point composed with a translation, which are equivalently rotations about some other point.

Since the elements in (1), (2), and (4) have finite order, every torsion-free subgroup of Δ must be contained in T . Hence $\Delta_e \leq T$, and therefore

$$[\Delta : \Delta_e] = [\Delta : T] \cdot [T : \Delta_e] = d \cdot [T : \Delta_e].$$

On the other hand, it is known that $[\Delta : \Delta_e]$ equals the number of edges of the dessin. Hence

$$[\Delta : \Delta_e] = md.$$

Combining these equalities yields

$$[T : \Delta_e] = m. \quad (5.5)$$

Using the results above, we obtain the following proposition.

Proposition 5.1. *A uniform passport of genus 1 admits a non-regular dessin if and only if it does not contain an element of type n^1 .*

In other words, the genus-1 uniform passports $[2^{2m}, 4^m, 4^m]$, $[3^m, 3^m, 3^m]$, $[3^{2m}, 2^{3m}, 6^m]$, and those obtained by permuting their entries admit a non-regular dessin if and only if $m \geq 2$.

Proof. When $m = 1$, (5.5) implies $\Delta_e = T$. Since $T \triangleleft \Delta$, by (5.1), the corresponding dessin is always regular.

When $m \geq 2$, (5.5) shows that Δ_e is a subgroup of T of index m . Thus, in each of (5.2), (5.3), and (5.4), we may take the subgroup T' obtained by replacing k with mk' ($k' \in \mathbb{Z}$), regard it as Δ_e , and consider the dessin $\mathcal{D}$ corresponding to this subgroup. In this situation, by (5.1) and the definition of a normal subgroup, we have

$$\mathcal{D} \text{ is regular} \iff T' \triangleleft \Delta \iff \text{for all } g \in \Delta \text{ and all } h \in T', ghg^{-1} \in T'.$$

Taking g to be Z (the rotation about the point C by angle $2\pi/c$) and taking h to be the element of T' obtained by setting $k = m$, $l = 1$ in each of (5.2), (5.3), and (5.4), we compute ghg^{-1} as follows:

- $\Delta = \Delta(2, 4, 4)$:

$$\begin{aligned} g &: (u, v) \mapsto (-v + 1, u + 1), \\ h &: (u, v) \mapsto (u + 2m, v + 2), \\ ghg^{-1} &: (u, v) \mapsto (u - 2, v + 2m) \end{aligned}$$

Since $m \geq 2$, $u - 2$ cannot be written as $u + 2mk'$ with $k' \in \mathbb{Z}$. Hence $ghg^{-1} \notin T'$.

- $\Delta = \Delta(3, 3, 3)$:

$$\begin{aligned} g &: (u, v) \mapsto \left(-\frac{1}{2}u - \frac{\sqrt{3}}{2}v + \frac{3}{2}, \frac{\sqrt{3}}{2}u - \frac{1}{2}v + \frac{\sqrt{3}}{2} \right), \\ h &: (u, v) \mapsto \left(u + \frac{3}{2}(m + 1), v + \frac{\sqrt{3}}{2}(m - 1) \right), \\ ghg^{-1} &: (u, v) \mapsto \left(u - \frac{3}{2}m, v + \frac{\sqrt{3}}{2}(m + 2) \right) \end{aligned}$$

Thus the corresponding translation parameters satisfy $k + l = -m$ and $k - l = m + 2$, hence $k = 1$. Since $m \geq 2$, this cannot be expressed in the form $k = mk'$ with $k' \in \mathbb{Z}$; therefore $ghg^{-1} \notin T'$.

- $\Delta = \Delta(2, 3, 6)$:

$$\begin{aligned} g: (u, v) &\mapsto \left(\frac{1}{2}u - \frac{\sqrt{3}}{2}v + \frac{3}{2}, \frac{\sqrt{3}}{2}u + \frac{1}{2}v + \frac{\sqrt{3}}{2} \right), \\ h: (u, v) &\mapsto \left(u + 3(m+1), v + \sqrt{3}(m-1) \right), \\ ghg^{-1}: (u, v) &\mapsto \left(u + 3, v + (2m+1)\sqrt{3} \right) \end{aligned}$$

Here the translation satisfies $k + l = 1$ and $k - l = 2m + 1$, hence $k = m + 1$. Since $m \geq 2$, this cannot be expressed in the form $k = mk'$ with $k' \in \mathbb{Z}$; therefore $ghg^{-1} \notin T'$.

In each case we conclude that T' is not normal in Δ . Hence, by (5.1), the corresponding dessin $\mathcal{D}$ is not regular. This proves that for every $m \geq 2$, for each of (5.2), (5.3), and (5.4), a non-regular dessin of genus 1 with the given uniform passport exists.

Combining all the above, we conclude that the existence of a non-regular dessin is equivalent to the condition $m \geq 2$. $\square$

For $m \geq 2$, a regular dessin does not necessarily exist. There are genus-1 passports for which only non-regular dessins exist; see Example 6.5 below.

For the automorphism group $\text{Aut } \mathcal{D}$ of a dessin $\mathcal{D}$, it is known that

$$\text{Aut } \mathcal{D} \cong N_{\Delta}(\Delta_e)/\Delta_e, \quad (5.6)$$

where

$$N_{\Delta}(\Delta_e) := \{g \in \Delta \mid g\Delta_e g^{-1} = \Delta_e\}$$

is the normalizer of Δ_e in Δ (see [JW16, Theorem 2.2]).

Using this, we obtain the following proposition.

Proposition 5.2. *For any uniform dessin d'enfant $\mathcal{D}$ of genus 1, we have $\text{Aut } \mathcal{D} \not\cong \{1\}$.*

Proof. As described above, the passports of genus-1 dessins are exactly $[2^{2m}, 4^m, 4^m]$, $[3^m, 3^m, 3^m]$, and $[2^{3m}, 3^{2m}, 6^m]$ together with those obtained by permuting their entries, and they satisfy $[T : \Delta_e] = m$.

For $m = 1$, Proposition 5.1 implies that $\mathcal{D}$ is regular. Therefore, by Proposition 2.13, $\text{Aut } \mathcal{D}$ is isomorphic to the monodromy group of $\mathcal{D}$. Moreover, by Lemma 2.10, the order of the monodromy group equals the number of edges of the dessin. Thus $|\text{Aut } \mathcal{D}| > 1$, that is, $\text{Aut } \mathcal{D} \not\cong \{1\}$.

For $m \geq 2$, since T commutes with every element of Δ , we have $g\Delta_e g^{-1} = \Delta_e$ for any $g \in T$. Hence

$$T \leq N_{\Delta}(\Delta_e).$$

Together with (5.6), this implies

$$\text{Aut } \mathcal{D} \cong N_{\Delta}(\Delta_e)/\Delta_e \geq T/\Delta_e.$$

Since $|T/\Delta_e| = m \geq 2$, it follows that $|\text{Aut } \mathcal{D}| \geq 2$, and therefore $\text{Aut } \mathcal{D} \not\cong \{1\}$.

Thus, for all $m \geq 1$, we have $\text{Aut } \mathcal{D} \not\cong \{1\}$. $\square$

6. Regularity Criteria in Arbitrary Genus

In this section, we describe two genus-independent criteria for the existence of regular dessins associated with uniform passports.

In what follows, let $\sigma_n := (1 \ 2 \ \dots \ n) \in S_n$.

6.1. Tree Cases

A passport $[\lambda_0, \lambda_1, \lambda_\infty]$ is called a *tree* if at least one of the λ_i is n^1 .

Theorem 6.1. *For a uniform passport $[a^p, b^q, n]$ ($p, q \geq 1$), there exists a regular dessin d'enfant with this passport if and only if $\gcd(p, q) = 1$.*

The same statement holds for the passports $[a^p, n, b^q]$ and $[n, a^p, b^q]$.

Proof. As regularity is symmetric with respect to black vertices, white vertices, and faces, it suffices to prove the statement for $[a^p, b^q, n]$ ($p, q \geq 1$).

By (2.4), the genus is $(n - (p + q) + 1)/2$. Hence $n - (p + q)$ is odd.

Since xy and $(xy)^{-1}$ have the same cycle type, it follows that xy is of cycle type (n) . Hence we may assume $xy = \sigma_n$. Since $\langle \sigma_n \rangle \cong C_n$ and $\langle \sigma_n \rangle \leq G$, and since a regular dessin satisfies $|G| = n$ by Lemma 2.10, it follows that if the dessin is regular, then $G = \langle \sigma_n \rangle \cong C_n$. Consequently, both x and y must lie in $\langle \sigma_n \rangle$.

Therefore,

$$\begin{aligned} \text{The dessin is regular} &\Leftrightarrow x, y \in \langle \sigma_n \rangle, \ x: \text{type } (a^p), \ y: \text{type } (b^q), \ xy = \sigma_n \\ &\Leftrightarrow \text{There exist } l, m \text{ such that } 1 \leq l < a, \ 1 \leq m < b, \\ &\quad x = \sigma_n^{lp}, \ \gcd(a, l) = 1, \ y = \sigma_n^{mq}, \ \gcd(b, m) = 1, \\ &\quad lp + mq \equiv 1 \pmod{n}. \end{aligned} \tag{6.1}$$

When the dessin is regular, since $lp + mq \equiv 1 \pmod{n}$ and $p, q \mid n$, it follows that $\gcd(p, q) = 1$.

Conversely, assume that $\gcd(p, q) = 1$. Since $n = pa = qb$, we have $q \mid a$ and $p \mid b$. Let $a = uq$ with $u \in \mathbb{Z}_{>0}$. Then we have $n = upq$ and $b = up$.

Moreover, since $\gcd(p, q) = 1$, there exist $\alpha, \beta \in \mathbb{Z}$ such that $p\alpha + q\beta = n + 1$. For these, we have

$$p(\alpha - kq) + q(\beta + kp) = n + 1 \quad \text{for all } k \in \mathbb{Z}.$$

Let k_0 be the smallest k such that $1 \leq \alpha - kq < a$ and $1 \leq \beta + kp < b$, and set $l_0 = \alpha - k_0q$, $m_0 = \beta + k_0p$. Then we obtain

$$\begin{aligned} l_0p + m_0q &= n + 1, \\ (l_0 - q)p + (m_0 + p)q &= n + 1, \\ &\dots \\ (l_0 - (u - 1)q)p + (m_0 + (u - 1)p)q &= n + 1, \end{aligned}$$

that is, u such equations.

The u pairs of coefficients $(l_0 - kq, m_0 + kp)$ ($0 \leq k \leq u - 1$) all satisfy $1 \leq l_0 - kq < a$ and $1 \leq m_0 + kp < b$.

To establish (6.1), it suffices to show that there exists some k_1 such that

$$\gcd(a, l_0 - k_1q) = \gcd(b, m_0 + k_1p) = 1, \quad 0 \leq k_1 \leq u - 1.$$

When $(l_0 - kq)p + (m_0 + kp)q = n + 1$, we have $(l_0 - kq)p \equiv 1 \pmod{q}$, and hence $\gcd(q, l_0 - kq) = 1$. Since $a = uq$, if $\gcd(u, l_0 - kq) = 1$, then $\gcd(a, l_0 - kq) = 1$ also holds.

Similarly, if $\gcd(u, m_0 + kp) = 1$, then we have $\gcd(b, m_0 + kp) = 1$.

Therefore, it suffices to show that there exists k such that

$$\gcd(u, l_0 - kq) = \gcd(u, m_0 + kp) = 1, \quad 0 \leq k \leq u - 1. \quad (6.2)$$

First, when $u = 1$, $k = 0$ satisfies (6.2).

When $u \geq 2$, let the prime factors of u be $p_1, \dots, p_s$ ($s \geq 1$, $p_1 < \dots < p_s$). Since $\gcd(p, q) = 1$, no p_i divides both p and q . Hence, we can choose $0 \leq k_1 \leq u - 1$ such that

$$\text{for all } 1 \leq i \leq s, \quad \begin{cases} k_1 \not\equiv l_0 q^{-1} \pmod{p_i} & (p_i \mid p, p_i \nmid q) \\ k_1 \not\equiv -m_0 p^{-1} \pmod{p_i} & (p_i \nmid p, p_i \mid q) \\ k_1 \not\equiv l_0 q^{-1}, -m_0 p^{-1} \pmod{p_i} & (p_i \nmid p, q) \end{cases} \quad (6.3)$$

Note that if $p_i = 2$, then u is even, and hence n is even. Since $n - (p + q)$ is odd, it follows that one of p or q must be even. Therefore, if $p_i \nmid p, q$, we must have $p_i \geq 3$, and hence a k_1 satisfying (6.3) can always be chosen.

For this k_1 and each p_i , we have:

- When $p_i \mid p$ and $p_i \nmid q$,
 $l_0 - k_1 q \not\equiv l_0 - l_0 q^{-1} q \equiv 0 \pmod{p_i}$,
and from $(l_0 - k_1 q)p + (m_0 + k_1 p)q = n + 1$, we obtain $(m_0 + k_1 p)q \equiv 1 \pmod{p_i}$,
hence $m_0 + k_1 p \not\equiv 0 \pmod{p_i}$.
- When $p_i \nmid p$ and $p_i \mid q$,
 $m_0 + k_1 p \not\equiv m_0 - m_0 p^{-1} p \equiv 0 \pmod{p_i}$,
and from $(l_0 - k_1 q)p + (m_0 + k_1 p)q = n + 1$, we obtain $(l_0 - k_1 q)p \equiv 1 \pmod{p_i}$, hence
 $l_0 - k_1 q \not\equiv 0 \pmod{p_i}$.
- When $p_i \nmid p, q$,
 $l_0 - k_1 q \not\equiv l_0 - l_0 q^{-1} q \equiv 0 \pmod{p_i}$,
 $m_0 + k_1 p \not\equiv m_0 - m_0 p^{-1} p \equiv 0 \pmod{p_i}$.

Hence, for every prime divisor p_i of u , we have $p_i \nmid l_0 - k_1 q$ and $p_i \nmid m_0 + k_1 p$, thus $k = k_1$ satisfies (6.2).

Therefore, when $\gcd(p, q) = 1$, a regular dessin exists. $\square$

In genus 0, the tree case corresponds to $[n, 1^n, n]$. Moreover, the following result holds from Theorem 6.1.

Corollary 6.2. *The passport $[n, b^q, n]$ always admits a regular dessin. On the other hand, the passport $[a^p, a^p, n]$ admits a regular dessin if and only if $p = 1$, equivalently $a = n$.*

6.2. General Criteria

The following holds regardless of whether the passport corresponds to a tree or not.

Proposition 6.3. *A uniform passport $[a^p, b^q, c^r]$, where $n = pa = qb = rc$, admits a regular dessin if and only if there exists an (abstract) group G of order n satisfying the following condition:*

- *There exist elements $x, y \in G$ such that $\text{ord}(x) = a$, $\text{ord}(y) = b$, $\text{ord}(xy) = c$, and $G = \langle x, y \rangle$.*

Proof. Assume that a uniform passport $[a^p, b^q, c^r]$ admits a regular dessin. As in Definition 2.6, we define permutations x and y acting on the set of edges of the dessin, and

set $G = \langle x, y \rangle$, the monodromy group. Then $\text{ord}(x) = a$, $\text{ord}(y) = b$, and $\text{ord}(xy) = \text{ord}((xy)^{-1}) = c$. By Lemma 2.10, we have $|G| = n$.

Conversely, suppose that a group G and elements $x, y \in G$ satisfy $|G| = n$, $\text{ord}(x) = a$, $\text{ord}(y) = b$, $\text{ord}(xy) = c$, and $G = \langle x, y \rangle$. Take an arbitrary bijection $\psi: E \rightarrow G$ on the set of edges $E = \{1, \dots, n\}$, and define an action of G on E by

$$\text{for all } g \in G \text{ and all } e \in E, \quad g \cdot e := \psi^{-1}(g\psi(e)).$$

For any $g, g' \in G$ and $e \in E$, we have

$$\begin{aligned} \text{id} \cdot e &= \psi^{-1}(\text{id} \psi(e)) = \psi^{-1}(\psi(e)) = e, \\ g' \cdot (g \cdot e) &= g' \cdot \psi^{-1}(g\psi(e)) = \psi^{-1}(g' \psi(\psi^{-1}(g\psi(e)))) = \psi^{-1}(g' g \psi(e)) = (g' g) \cdot e. \end{aligned}$$

Therefore, this defines a group action of G on E .

Moreover, for any $e_1, e_2 \in E$, consider the element $\psi(e_2)(\psi(e_1))^{-1} \in G$. Then

$$\psi(e_2)(\psi(e_1))^{-1} \cdot e_1 = \psi^{-1}(\psi(e_2)(\psi(e_1))^{-1} \psi(e_1)) = \psi^{-1}(\psi(e_2)) = e_2.$$

Hence this action is transitive.

Therefore, there exists a dessin whose monodromy group is G [Sco24, Theorem 3.6]. Since $|G| = n$, it follows from Lemma 2.10 that the dessin is regular. Moreover, the orders of x , y , and xy are a , b , and c , respectively, and since the dessin is uniform by Proposition 2.15, its passport is $[a^p, b^q, c^r]$ with $n = pa = qb = rc$. $\square$

Example 6.4. A regular dessin d'enfant with the passport $[3^4, 3^4, 3^4]$ ($n = 12$, genus 1) must satisfy $\text{ord}(x) = \text{ord}(y) = \text{ord}(xy) = 3$.

Furthermore, since the action of $G = \langle x, y \rangle$ must be transitive, we have $x \neq y$. Also, as $x, y \neq 1$, it follows that $xy \neq x$ and $xy \neq y$. Hence x , y , and xy must all be distinct, and therefore G must be of order 12 and contain at least three elements of order 3.

There are five abstract groups of order 12: the cyclic group C_{12} , the direct product $C_6 \times C_2$, the alternating group A_4 , the dihedral group D_6 , and the dicyclic group Q_{12} . Among these, only A_4 contains three or more elements of order 3 (in fact, four). By examining the elements of A_4 , for example, we find that

$$\begin{aligned} x &= (1 \ 2 \ 3)(4 \ 5 \ 6)(7 \ 8 \ 9)(10 \ 11 \ 12), \\ y &= (1 \ 4 \ 7)(2 \ 11 \ 6)(3 \ 8 \ 10)(5 \ 12 \ 9), \\ xy &= (1 \ 5 \ 10)(2 \ 12 \ 7)(3 \ 9 \ 6)(4 \ 8 \ 11) \end{aligned}$$

satisfy the required conditions, showing that a regular dessin exists in this passport.

Example 6.5. A regular dessin d'enfant with the passport $[3^5, 3^5, 3^5]$ ($n = 15$, genus 1) must satisfy $\text{ord}(x) = \text{ord}(y) = \text{ord}(xy) = 3$. Since the only group of order 15 is the cyclic group C_{15} , if σ is a generator of C_{15} , then x , y , and xy must each be one of σ^5 or σ^{10} . However, $\{\sigma^5, \sigma^{10}\}$ does not generate C_{15} .

Therefore, no regular dessin exists for the passport $[3^5, 3^5, 3^5]$.

By the same reasoning, no regular dessin exists for the passport $[5^3, 5^3, 5^3]$ ($n = 15$, genus 4) either.

7. Automorphism Groups in Genus ≥ 2

In this section, we prove the existence of a dessin $\mathcal{D}$ with $\text{Aut } \mathcal{D} \cong \{1\}$ for passports of genus at least 2 in which at least two of the entries are of type n^1 .

7.1. Passports $[n, n, n]$

Proposition 7.1. *Every passport $[n, n, n]$ of genus ≥ 2 admits a dessin with a trivial automorphism group.*

Proof. By (2.4), the genus is $(n-3)/2 + 1 = (n-1)/2$. Since it is an integer at least 2, n is odd and $n \geq 5$.

Since the monodromy group does not depend on the labeling of the edges, we may fix $x = (1 \ 2 \ \dots \ n)$.

Let

$$y = (2 \ 4 \ 6 \ \dots \ n-1 \ n \ n-2 \ \dots \ 3 \ 1),$$

where the even numbers appear in ascending order, followed by the odd numbers in descending order. Then

$$xy = (1 \ 3 \ 2 \ 5 \ 4 \ \dots \ n-2 \ n-3 \ n \ n-1),$$

which is of type (n) .

For any $i, j \in E = \{1, \dots, n\}$, we have $x^{j-i} \cdot i = j$. Hence, $G = \langle x, y \rangle$ acts transitively on E .

Therefore, G is the monodromy group of a dessin $\mathcal{D}$ with passport $[n, n, n]$. Since n is odd, we have $G \leq A_n$, the alternating group.

Computing $x^2(xyxy)^{-1}$ yields $(1 \ 2 \ 3)$. As is well known, the permutations x and $(1 \ 2 \ 3)$ generate A_n for odd $n \geq 5$, and hence $G = A_n$. Therefore,

$$\text{Aut } \mathcal{D} \cong C_{S_n}(A_n) \cong \{1\}.$$

□

7.2. Approach for the Passports $[n, b^q, n]$ ($q \geq 2$)

In this section, we describe our approach to proving that every uniform passport $[a^p, b^q, c^r]$ of genus at least 2, with exactly two of p, q, r equal to 1, admits a dessin d'enfant with trivial automorphism group. The proof combines counting arguments, analytic estimates, and a finite verification of exceptional cases.

Since the automorphism group is invariant under interchanging black vertices, white vertices, and faces, it suffices to consider the case $[n, b^q, n]$, where $n = bq$ and $q \geq 2$. Since the genus $(n-q)/2$ is an integer at least 2, we have $n-q \geq 4$, $n \equiv q \pmod{2}$, and hence $b = n/q \geq 2$.

In this case, unlike in Proposition 7.1, it is difficult to construct a general pair (x, y) such that $\text{Aut } \mathcal{D} \cong \{1\}$. Therefore, we exploit the relationship between the primitivity (see Definition 2.17 and 2.18) of the monodromy group and the triviality of the automorphism group.

Proposition 7.2. *Let $G = \langle x, y \rangle$ be the monodromy group of a dessin $\mathcal{D}$ with n edges. If n is not prime and G is primitive, the automorphism group $\text{Aut } \mathcal{D}$ is trivial.*

Proof. Assume that n is not prime, G is primitive, and let $c \in \text{Aut } \mathcal{D}$. Then, for every $g \in G$, we have $cg = gc$.

For $e \in \{1, \dots, n\}$, let $O_e = \langle c \rangle \cdot e$ denote the orbit of e under the action of $\langle c \rangle$. For any $g \in G$, we have

$$g \cdot O_e = g \cdot (\langle c \rangle \cdot e) = (g\langle c \rangle) \cdot e = (\langle c \rangle g) \cdot e = \langle c \rangle \cdot (g \cdot e) = O_{g \cdot e}.$$

Assume that $(g \cdot O_e) \cap O_e \neq \emptyset$. Since $g \cdot O_e = O_{g \cdot e}$ and each element of O_e can be written as $c^m \cdot e$, while each element of $O_{g \cdot e}$ can be written as $c^l g \cdot e$, there exist integers l, m such that $c^l g \cdot e = c^m \cdot e$. Then $g \cdot e = c^{m-l} \cdot e \in O_e$, and hence $g \cdot O_e \subset O_e$. Similarly, since $e = c^{l-m} g \cdot e \in O_{g \cdot e}$, we have $O_e \subset O_{g \cdot e}$. Therefore, $g \cdot O_e = O_{g \cdot e} = O_e$.

Since this holds for all g and all e , we have, for every g and e , that either $g \cdot O_e = O_e$ or $(g \cdot O_e) \cap O_e = \emptyset$. Hence, each element of the family $\{O_e\}$ is a block for the action of G .

Furthermore, since G acts transitively, all blocks are mapped to one another under the action of G and therefore have the same size d . Since the size of the orbit of c is d , the cycle type of c is $(d^{n/d})$.

If $1 < d < n$, then G has nontrivial blocks of size d , contradicting the assumption that G is primitive.

If $d = n$, then c has cycle type (n) . It is well known that $C_{S_n}(c) = \langle c \rangle$. Since $cg = gc$ for every $g \in G$, it follows that $G \leq \langle c \rangle$, and hence G is cyclic. A cyclic group acts primitively only when n is prime; since n is not prime, this yields a contradiction.

Consequently, we have $d = 1$, and the cycle type of c is (1^n) ; that is, $c = \text{id}$. Therefore, $\text{Aut } \mathcal{D} \cong \{1\}$. $\square$

For the passport $[n, b^q, n]$ with $b \geq 2$, fix $x = (1 \ 2 \ \dots \ n) \in S_n$. As described in the proof of Proposition 7.1, the group $G = \langle x, y \rangle$ acts transitively on the edges $\{1, \dots, n\}$ for any $y \in S_n$. Therefore, if y has cycle type (b^q) and xy has cycle type (n) , then G is the monodromy group of a dessin corresponding to this passport.

We define the following subsets of S_n :

- $T(b, q)$: the set of elements of S_n with cycle type (b^q) ;
- $N(b, q)$: the set of elements $y \in T(b, q)$ such that xy has cycle type (n) ;
- $I(b, q)$: the set of elements $y \in T(b, q)$ such that $G = \langle x, y \rangle$ is imprimitive;
- $C(b, q)$: the set of elements $y \in T(b, q)$ such that $C_{S_n}(G)$ is nontrivial (equivalently, $\text{Aut } \mathcal{D} \not\cong \{1\}$).

T , N , and I are the same as those defined in Section 3.1.

Since $n = bq$ is not prime, it follows from Proposition 7.2 that $I \supset C$.

The existence of a dessin $\mathcal{D}$ such that $\text{Aut } \mathcal{D} \cong \{1\}$ is equivalent to $N \cap C^c \neq \emptyset$ (the gray region in Figure 8). To prove this, however, it is sufficient to show that $N \cap I^c \neq \emptyset$. Furthermore, to establish this, it suffices to show that $|N| > |I|$.

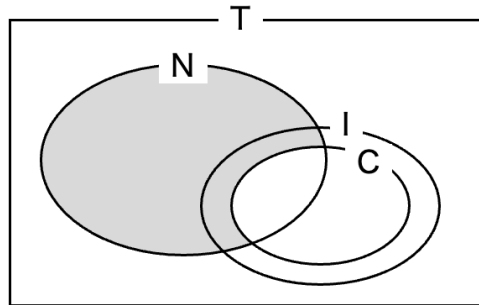


FIGURE 8. The region corresponding to trivial automorphism groups

Hereafter, we use the same notation to denote the number of elements in a set (for example, the number of elements of $T(b, q)$ is denoted by the function $T(b, q)$).

For a passport $[n, b^q, n]$ ($b \geq 2$) of genus at least 2, we show that

$$\frac{I(b, q)}{T(b, q)} < \frac{N(b, q)}{T(b, q)}, \quad (7.1)$$

which implies that there always exists a dessin $\mathcal{D}$ whose automorphism group $\text{Aut } \mathcal{D}$ is trivial.

By directly computing the automorphism groups of dessins belonging to the corresponding passports, we find that most of them are in fact trivial, and that this tendency becomes more pronounced as n increases. Accordingly, we allow a slightly coarse upper bound for I/T , and in the remaining cases where (7.1) cannot be established for finitely many small pairs (b, q) , we complete the proof by explicitly exhibiting pairs (x, y) for which the automorphism group is trivial.

As in Section 3.1, let $I_m(b, q)$, where $m \mid n$ and $2 \leq m < n$, denote the subset of $I(b, q)$ consisting of elements y such that $G = \langle x, y \rangle$ is imprimitive with respect to the residue classes modulo m .

Proposition 3.1 and Theorem 3.5 give an explicit formula for I_m/T and provide an upper bound for I/T :

$$\frac{I_m(b, q)}{T(b, q)} = \frac{q!}{n!} m! \left(\left(\frac{n}{m} \right)! \right)^m \sum_{\{(d_i, t_i)\}} \prod_i \frac{1}{d_i^{t_i} t_i!} \left(\frac{d_i^{d_i q}}{\left(\frac{d_i q}{m} \right)!} \right)^{t_i}, \quad (7.2)$$

$$\frac{I(b, q)}{T(b, q)} \leq \frac{q!}{n!} \sum_{\substack{2 \leq m < n \\ m \mid n}} m! \left(\left(\frac{n}{m} \right)! \right)^m \sum_{\{(d_i, t_i)\}} \prod_i \frac{1}{d_i^{t_i} t_i!} \left(\frac{d_i^{d_i q}}{\left(\frac{d_i q}{m} \right)!} \right)^{t_i}, \quad (7.3)$$

where the summations over $\{(d_i, t_i)\}$ are taken over all partitions $\{(d_i, t_i)\}$ of m satisfying (3.9), that is:

$$\sum_i d_i t_i = m, \quad d_i \text{ are all distinct}, \quad d_i \mid b, \quad m \mid d_i q \text{ for all } i.$$

Since there exist pairs (x, y) that admit blocks corresponding to multiple values of m , the total sum on the right-hand side of the latter generally exceeds I/T .

Since

$$\sum_i t_i \frac{d_i q}{m} = \frac{q}{m} \sum_i d_i t_i = q,$$

it follows from the logarithmic convexity of the factorial (equivalently, Gamma function) that the sum of the terms $\log(d_i q/m)$, each counted with multiplicity t_i , is not smaller than the sum of m terms of $\log(q/m)$, where m is the total number of terms. Hence,

$$\log \prod_i \left(\left(\frac{d_i q}{m} \right)! \right)^{t_i} = \sum_i t_i \log \left(\frac{d_i q}{m} \right)! \geq m \log \left(\frac{q}{m} \right)!$$

where, since q/m is not necessarily an integer, we define $(q/m)! := \Gamma(q/m + 1)$.

Therefore,

$$\prod_i \left(\frac{1}{\left(\frac{d_i q}{m} \right)!} \right)^{t_i} \leq \frac{1}{\left(\left(\frac{q}{m} \right)! \right)^m}.$$

This yields

$$\frac{I_m(b, q)}{T(b, q)} \leq \frac{q!}{n!} m! \left(\frac{\left(\frac{n}{m}\right)!}{\left(\frac{q}{m}\right)!} \right)^m \sum_{\{(d_i, t_i)\}} \prod_i \frac{d_i^{\left(\frac{d_i q}{m} - 1\right)t_i}}{t_i!}.$$

Applying Stirling's formula

$$\sqrt{2\pi M} \left(\frac{M}{e} \right)^M e^{\frac{1}{12M+1}} \leq M! \leq \sqrt{2\pi M} \left(\frac{M}{e} \right)^M e^{\frac{1}{12M}} \quad (M > 0), \quad (7.4)$$

we obtain

$$\begin{aligned} \frac{q!}{n!} m! \left(\frac{\left(\frac{n}{m}\right)!}{\left(\frac{q}{m}\right)!} \right)^m &\leq m! \cdot \frac{\sqrt{2\pi q} \left(\frac{q}{e}\right)^q \exp\left(\frac{1}{12q}\right)}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n+1}\right)} \cdot \frac{(\sqrt{2\pi \frac{n}{m}})^m \left(\frac{n}{me}\right)^n \exp\left(\frac{m}{12\frac{n}{m}}\right)}{(\sqrt{2\pi \frac{q}{m}})^m \left(\frac{q}{me}\right)^q \exp\left(\frac{m}{12\frac{q}{m}+1}\right)} \\ &= m! b^{\frac{m-1}{2}} m^{-q(b-1)} \exp\left(\frac{1}{12q} + \frac{m^2}{12n} - \frac{1}{12n+1} - \frac{m^2}{12q+m}\right). \end{aligned}$$

If we regard the exponent of $\exp$ as a function of m ($2 \leq m \leq n$), denoted by $f(m)$, then

$$\begin{aligned} f(m) &= \frac{1}{12q} + \frac{m^2}{12n} - \frac{1}{12n+1} - \frac{m^2}{12q+m}, \\ f'(m) &= m \left(\frac{1}{6n} - \frac{m+24q}{(m+12q)^2} \right). \end{aligned}$$

Let $g(m) = -(m+24q)/(m+12q)^2$. Then $g'(m) = (m+36q)/(m+12q)^3 > 0$, thus $g(m)$ is monotonically increasing. Hence,

$$\begin{aligned} f'(m) &\leq f'(n) = n \left(\frac{1}{6n} - \frac{n+24q}{(n+12q)^2} \right) = bq \left(\frac{1}{6bq} - \frac{bq+24q}{(bq+12q)^2} \right) \\ &= b \left(\frac{1}{6b} - \frac{b+24}{(b+12)^2} \right) = -\frac{5b^2+120b-144}{6(b+12)^2} < 0, \end{aligned}$$

where the last inequality follows from $b \geq 2$.

Therefore, since $f(m)$ is monotonically decreasing, we have

$$\begin{aligned} f(m) &\leq f(2) = \frac{1}{12q} + \frac{1}{3n} - \frac{1}{12n+1} - \frac{2}{6q+1} < \frac{1}{12q} + \frac{1}{3n} - \frac{2}{6q+1} \\ &= \frac{1}{12q} + \frac{1}{3bq} - \frac{2}{6q+1} = \frac{-18bq+b+24q+4}{12bq(6q+1)} = -\frac{18(b-\frac{4}{3})(q-\frac{1}{18})-\frac{16}{3}}{12bq(6q+1)} \\ &\leq -\frac{18(2-\frac{4}{3})(2-\frac{1}{18})-\frac{16}{3}}{12bq(6q+1)} < 0, \end{aligned}$$

where the last inequality follows from $b, q \geq 2$.

Hence, $\exp(f(m)) < 1$, and therefore

$$\frac{q!}{n!} m! \left(\frac{\left(\frac{n}{m}\right)!}{\left(\frac{q}{m}\right)!} \right)^m \leq m! b^{\frac{m-1}{2}} m^{-q(b-1)},$$

and then,

$$\frac{I_m(b, q)}{T(b, q)} \leq m! b^{\frac{m-1}{2}} m^{-q(b-1)} S(b, q, m), \quad S(b, q, m) := \sum_{\{(d_i, t_i)\}} \prod_i \frac{d_i^{\left(\frac{d_i q}{m} - 1\right)t_i}}{t_i!}, \quad (7.5)$$

where the summation in $S(b, q, m)$ is taken over all partitions $\{(d_i, t_i)\}$ of m satisfying (3.9).

Applying (7.4) to $m!$ as well, we obtain

$$\begin{aligned} \frac{I_m(b, q)}{T(b, q)} &\leq \sqrt{2\pi m} \left(\frac{m}{e}\right)^m e^{\frac{1}{12m}} b^{\frac{m-1}{2}} m^{-q(b-1)} S(b, q, m) \\ &= \sqrt{2\pi} b^{\frac{m-1}{2}} m^{m-q(b-1)+\frac{1}{2}} e^{-m+\frac{1}{12m}} S(b, q, m). \end{aligned} \quad (7.6)$$

In the following, we prove (7.1) by dividing the argument into two cases: $b = 2$, where the dessins are called *clean* and which turns out to be the critical case, and $b \geq 3$.

7.3. The Case $b = 2$

Proposition 7.3. *Every passport $[n, 2^q, n]$ ($n = 2q$) of genus ≥ 2 admits a dessin with a trivial automorphism group.*

Proof. By (2.4), the genus is

$$\frac{2q - (1 + q + 1)}{2} + 1 = \frac{q}{2} \geq 2.$$

Therefore, q must be an even integer with $q \geq 4$.

By Theorem 3.3,

$$\frac{N}{T} = \frac{2}{n+2} = \frac{2}{2q+2} = \frac{1}{q\left(1+\frac{1}{q}\right)} \geq \frac{1}{q\left(1+\frac{1}{4}\right)} = \frac{4}{5q}.$$

Therefore, it suffices to show that

$$\frac{I(b, q)}{T(b, q)} < \frac{4}{5q}.$$

Let $\tau(n)$ denote the number of positive divisors of n . Then the number of values of m appearing in (7.3) is $\tau(n) - 2$. Since $\tau(n) \leq 2\sqrt{n} = 2\sqrt{2q}$, it suffices to show that for each m with $m \mid n$ and $2 \leq m < n$,

$$\frac{I_m}{T} < \frac{4}{5q} \cdot \frac{1}{2\sqrt{2q}} = \frac{\sqrt{2}}{5q^{\frac{3}{2}}}.$$

Since $n = 2q$, the largest divisor m of n with $m < n$ is $m = q$. Therefore, it suffices to show that

$$\log\left(\frac{5}{\sqrt{2}} q^{\frac{3}{2}} \frac{I_m}{T}\right) < 0 \quad (7.7)$$

holds for all $2 \leq m \leq q$.

Since $d_i \mid 2$ in (3.9), the only possible values of d_i are 1 and 2.

We now divide the analysis into the cases $m = 2$ and $3 \leq m \leq q$, and obtain an upper bound for the left-hand side of (7.7) in each case.

(i) The case $m = 2$.

Since q is even, the partitions $\{(d_i, t_i)\}$ of 2 satisfying (3.9) are $\{(1, 2)\}$ and $\{(2, 1)\}$. Hence, by (7.2),

$$\frac{I_2}{T} = \frac{q!}{(2q)!} 2!(q!)^2 \left(\frac{1}{2} \left(\frac{1}{\left(\frac{q}{2}\right)!} \right)^2 + \frac{1}{2} \cdot \frac{2^q}{q!} \right) = \frac{(q!)^2}{(2q)!} \left(\frac{q!}{\left(\left(\frac{q}{2}\right)!\right)^2} + 2^q \right).$$

Using Stirling's formula (7.4), we obtain

$$\begin{aligned}
\frac{I_2}{T} &\leq \frac{2\pi q \left(\frac{q}{e}\right)^{2q} e^{\frac{1}{6q}}}{\sqrt{2\pi} \cdot 2q \left(\frac{2q}{e}\right)^{2q} e^{\frac{1}{24q+1}}} \left(\frac{\sqrt{2\pi q} \left(\frac{q}{e}\right)^q e^{\frac{1}{12q}}}{2\pi \cdot \frac{q}{2} \left(\frac{q}{2e}\right)^q e^{\frac{2}{6q+1}}} + 2^q \right) \\
&= \frac{\sqrt{\pi q}}{2^q} \exp\left(\frac{1}{6q} - \frac{1}{24q+1}\right) \left(\sqrt{\frac{2}{\pi q}} \exp\left(\frac{1}{12q} - \frac{2}{6q+1}\right) + 1 \right).
\end{aligned}$$

Here, the first exponential term is decreasing in q , hence we bound it by its value at $q = 4$. The argument of the second exponential term is always negative, and thus we bound it by $e^0 = 1$. Hence,

$$\frac{I_2}{T} \leq \frac{e^{\frac{73}{2328}}}{2^q} (\sqrt{2} + \sqrt{\pi q}).$$

Therefore,

$$\log\left(\frac{5}{\sqrt{2}} q^{\frac{3}{2}} \frac{I_2}{T}\right) \leq \frac{73}{2328} - q \log 2 + \log(\sqrt{2} + \sqrt{\pi q}) + \log 5 - \frac{1}{2} \log 2 + \frac{3}{2} \log q. \quad (7.8)$$

Differentiating the right-hand side with respect to q gives

$$-\log 2 + \frac{1}{2} \sqrt{\frac{\pi}{q}} \frac{1}{\sqrt{2} + \sqrt{\pi q}} + \frac{3}{2q} \leq -\log 2 + \frac{1}{2} \sqrt{\frac{\pi}{4}} \frac{1}{\sqrt{2} + \sqrt{4\pi}} + \frac{3}{2 \cdot 4} < 0 \quad (q \geq 4).$$

Therefore, the right-hand side of (7.8) is decreasing for all $q \geq 4$, and a direct evaluation shows that it becomes negative for $q \geq 10$.

(ii) The case $3 \leq m \leq q$.

Substituting $b = 2$ into (7.6), we obtain

$$\frac{I_m}{T} \leq 2^{\frac{m}{2}} \sqrt{\pi} m^{m-q+\frac{1}{2}} e^{-m+\frac{1}{12m}} S(2, q, m). \quad (7.9)$$

Regarding (3.9), d_i must be 1 or 2 since $d_i \mid 2$. For $d_i = 1$, the condition $m \mid d_i q$ holds only if $m \mid q$. For $d_i = 2$, the condition $m \mid d_i q$ always holds, since $m \mid n (= 2q)$. Furthermore, if $m \nmid q$, then m must be even. In this case, the unique partition satisfying (3.9) is $\{(2, m/2)\}$.

Therefore, the partitions of m satisfying (3.9) are

$$\begin{aligned}
&\left\{ \{(1, m-2j), (2, j)\} \mid 0 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor \right\} \quad (m \mid q), \\
&\left\{ \left\{ \left(2, \frac{m}{2}\right) \right\} \right\} \quad (m \nmid q).
\end{aligned}$$

As the second set is contained in the first, in (7.9) we obtain the inequality

$$S(2, q, m) = \sum_{\{(d_i, t_i)\}} \prod_i \frac{d_i^{\binom{d_i q}{m} - 1} t_i}{t_i!} \leq \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{2^{\binom{2q}{m} - 1} j}{(m-2j)! j!}. \quad (7.10)$$

Let $\alpha = 2^{2q/m-1}$. Then

$$e^{x+\alpha x^2} = e^x e^{\alpha x^2} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{\infty} \frac{\alpha^j x^{2j}}{j!} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\alpha^j x^{k+2j}}{k! j!}.$$

Extracting the coefficient of x^m on the right-hand side, only the terms satisfying $k+2j = m$ (i.e. $k = m - 2j$) contribute. Hence we obtain

$$[x^m] e^{x+\alpha x^2} = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\alpha^j}{(m-2j)! j!},$$

where the left-hand side denotes the coefficient of x^m in $e^{x+\alpha x^2}$.

Since the right-hand side of (7.10) coincides with this expression, we may bound $S(2, q, m)$ using the generating function $e^{x+\alpha x^2}$ as

$$S(2, q, m) \leq [x^m] e^{x+\alpha x^2}.$$

Let $F(x) = e^{x+\alpha x^2}$. By Cauchy's integral formula, we have

$$S(2, q, m) \leq [x^m] F(x) = \frac{1}{2\pi i} \int_{|z|=R} \frac{F(z)}{z^{m+1}} dz \quad (R > 0).$$

Hence, for any $R > 0$,

$$S(2, q, m) = |S(2, q, m)| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{\max_{|z|=R} |F(z)|}{R^{m+1}} = \frac{\max_{|z|=R} |F(z)|}{R^m}.$$

For fixed $|z| = R$, the quantity $|F(z)|$ attains its maximum at $z = R$ (a positive real number). Thus,

$$S(2, q, m) \leq \frac{e^{R+\alpha R^2}}{R^m} \quad \text{for all } R > 0. \quad (7.11)$$

Viewing the right-hand side as a function of R , let $L(R)$ denote its logarithm. Then

$$\begin{aligned} L(R) &= R + \alpha R^2 - m \log R, \\ L'(R) &= 1 + 2\alpha R - \frac{m}{R} = \frac{1}{R}(2\alpha R^2 + R - m). \end{aligned}$$

Since the right-hand side of (7.11) (the upper bound for $S(2, q, m)$) is minimized when $L'(R) = 0$, i.e. at $R = (\sqrt{1 + 8\alpha m} - 1)/(4\alpha)$, we choose a nearby value

$$R = R_0 := \frac{\sqrt{8\alpha m}}{4\alpha} = \sqrt{\frac{m}{2\alpha}} \quad (> 0).$$

Then

$$\begin{aligned} S(2, q, m) &\leq \frac{e^{R_0+\alpha R_0^2}}{R_0^m} = \exp\left(\frac{m}{2} + \sqrt{\frac{m}{2\alpha}}\right) \left(\frac{2\alpha}{m}\right)^{\frac{m}{2}} \\ &= \exp\left(\frac{m}{2} + \sqrt{\frac{m}{2\alpha}}\right) \left(\frac{2\alpha}{m}\right)^{\frac{m}{2}} = 2^q m^{-\frac{m}{2}} \exp\left(\frac{m}{2} + 2^{-\frac{q}{m}} \sqrt{m}\right). \end{aligned} \quad (7.12)$$

From (7.5), (7.9), and (7.12), we have

$$\begin{aligned} \frac{5}{\sqrt{2}} q^{\frac{3}{2}} \frac{I_m}{T} &\leq \frac{5}{\sqrt{2}} q^{\frac{3}{2}} 2^{\frac{m}{2}+q} \sqrt{\pi m}^{\frac{m+1}{2}-q} \exp\left(-\frac{m}{2} + \frac{1}{12m} + 2^{-\frac{q}{m}} \sqrt{m}\right), \\ \log\left(\frac{5}{\sqrt{2}} q^{\frac{3}{2}} \frac{I_m}{T}\right) &\leq \frac{3}{2} \log q + \log 5 + \left(\frac{m}{2} + q - \frac{1}{2}\right) \log 2 + \frac{1}{2} \log \pi \\ &\quad + \left(\frac{m+1}{2} - q\right) \log m - \frac{m}{2} + \frac{1}{12m} + 2^{-\frac{q}{m}} \sqrt{m}. \end{aligned}$$

Let the right-hand side be denoted by $G(q, m)$. For $q \geq 4$ and $3 \leq m \leq q$, we have

$$\frac{\partial^2 G}{\partial m^2} = \frac{1}{2m} + \frac{2q-1}{2m^2} + \frac{1}{6m^3} + \frac{2^{-\frac{q}{m}}}{m^{\frac{7}{2}}} \left(-\frac{m^2}{4} - qm \log 2 + (q \log 2)^2\right). \quad (7.13)$$

Let

$$g(q, m) = -\frac{m^2}{4} - qm \log 2 + (q \log 2)^2.$$

Since $q \geq m$, we have

$$\frac{\partial g}{\partial q} = 2(\log 2)^2 q - m \log 2 \geq 2(\log 2)^2 m - m \log 2 = m \log 2(2 \log 2 - 1) > 0.$$

Hence g is increasing in q for each fixed m , and therefore

$$g(q, m) \geq g(m, m) = -\frac{m^2}{4} - m^2 \log 2 + (m \log 2)^2 = -Cm^2,$$

$$C := \log 2 + \frac{1}{4} - (\log 2)^2 > 0.$$

Therefore, in (7.13) we have

$$\frac{2^{-\frac{q}{m}}}{m^{\frac{7}{2}}} \left(-\frac{m^2}{4} - qm \log 2 + (q \log 2)^2 \right) \geq -\frac{2^{-\frac{q}{m}}}{m^{\frac{7}{2}}} Cm^2 \geq -\frac{2^{-\frac{m}{m}}}{m^{\frac{7}{2}}} Cm^2 = -\frac{C}{2m^{\frac{3}{2}}},$$

and it follows that

$$\begin{aligned} \frac{\partial^2 G}{\partial m^2} &\geq \frac{1}{2m} + \frac{2q-1}{2m^2} + \frac{1}{6m^3} - \frac{C}{2m^{\frac{3}{2}}} \geq \frac{1}{2m} + \frac{2m-1}{2m^2} + \frac{1}{6m^3} - \frac{C}{2m^{\frac{3}{2}}} \\ &= \frac{3}{2m} - \frac{1}{2m^2} + \frac{1}{6m^3} - \frac{C}{2m^{\frac{3}{2}}} > \frac{3}{2m} - \frac{1}{2m^2} - \frac{C}{2m^{\frac{3}{2}}} = \frac{1}{2m^{\frac{3}{2}}} \left(3\sqrt{m} - \frac{1}{\sqrt{m}} - C \right) \\ &\geq \frac{1}{2m^{\frac{3}{2}}} \left(3\sqrt{3} - \frac{1}{\sqrt{3}} - \log 2 - \frac{1}{4} + (\log 2)^2 \right) > 0. \end{aligned}$$

Therefore, G is convex in m . Hence its maximum on the interval $3 \leq m \leq q$ is attained at one of the endpoints, $m = 3$ or $m = q$.

Let $H_1(q) := G(q, 3)$ and $H_2(q) := G(q, q)$. Then

$$G(q, m) \leq \max\{H_1(q), H_2(q)\} \quad (3 \leq m \leq q),$$

where

$$H_1(q) = \frac{3}{2} \log q + \log 5 + (q+1) \log 2 + \frac{1}{2} \log \pi + (2-q) \log 3 - \frac{53}{36} + 2^{-\frac{q}{3}} \sqrt{3},$$

$$H'_1(q) = \frac{3}{2q} + \log 2 - \log 3 - \frac{2^{-\frac{q}{3}}}{\sqrt{3}} \log 2 < \frac{3}{2 \cdot 4} + \log 2 - \log 3 < 0 \quad (q \geq 4),$$

and

$$H_2(q) = \log 5 + \left(\frac{3}{2}q - \frac{1}{2} \right) \log 2 + \frac{1}{2} \log \pi + \left(-\frac{q}{2} + 2 \right) \log q - \frac{q}{2} + \frac{1}{12q} + \frac{1}{2} \sqrt{q},$$

$$H'_2(q) = -\frac{1}{2} \log q - 1 + \frac{3}{2} \log 2 + \frac{2}{q} - \frac{1}{12q^2} + \frac{1}{4\sqrt{q}}$$

$$< -\frac{1}{2} \log 4 - 1 + \frac{3}{2} \log 2 + \frac{2}{4} - 0 + \frac{1}{4\sqrt{4}} < 0 \quad (q \geq 4).$$

Thus both H_1 and H_2 are strictly decreasing for $q \geq 4$. Direct computation shows that $H_1(q) < 0$ for $q \geq 22$, and $H_2(q) < 0$ for $q \geq 14$. Consequently, $G(q, m)$ is negative for all $q \geq 22$ and all m with $3 \leq m \leq q$.

Combining this with the case $m = 2$, we see that (7.7) holds for all $q \geq 22$ and all m with $m \mid n$, $2 \leq m < n$.

For even integers $4 \leq q \leq 20$, Table 2 lists those permutations y such that, with $x = (1 \ 2 \ \dots \ n)$ fixed, the automorphism group is trivial.

When $q = 4$, the monodromy group $G = \langle x, y \rangle$ has order 336 and $C_{S_n}(G) \cong \{1\}$. On the other hand, when $6 \leq q \leq 20$, we have $\langle x, y \rangle = S_n$. Since $C_{S_n}(S_n) = \{1\}$, it follows that for each such q there exists a dessin $\mathcal{D}$ with $\text{Aut } \mathcal{D} \cong \{1\}$.

Consequently, every passport $[n, 2^q, n]$ ($n = 2q$) of genus ≥ 2 admits a dessin with a trivial automorphism group. $\square$

b	q	y	G	$w(x, y)$	p_w	w
2	4	(1 4)(2 5)(3 7)(6 8)	336			
2	6	(1 4)(2 9)(3 6)(5 8)(7 11)(10 12)	S_{12}	$xyxyx^4yx^3yx$	5	(4 6 9 5 11)
2	8	(1 13)(2 10)(3 16)(4 6)(5 9)(7 14)(8 11)(12 15)	S_{16}	$x^3yxyxyx^3$	13	(1 15 8 7 16 10 11 2 3 5 6 12 13)
2	10	(1 15)(2 19)(3 12)(4 18)(5 8)(6 16)(7 10)(9 11)(13 20)(14 17)	S_{20}	yx^4yxyx^2	17	(1 10 8 3 4 15 5 6 12 9 17 2 11 19 20 13 16)
2	12	(1 16)(2 11)(3 9)(4 14)(5 20)(6 21)(7 15)(8 17)(10 24)(12 22)(13 18)(19 23)	S_{24}	$yx^2yx^4yx^2yx^4$	19	(1 9 20 18 23 19 11 8 21 24 12 5 22 4 2 6 3 10 13)
2	14	(1 27)(2 18)(3 20)(4 8)(5 22)(6 13)(7 26)(9 12)(10 15)(11 28)(14 17)(16 23)(19 25)(21 24)	S_{28}	$x^3yx^{11}y$	23	(1 18 9 19 7 15 27 12 6 24 11 3 23 4 28 8 13 17 22 26 5 25 21)
2	16	(1 26)(2 27)(3 29)(4 15)(5 20)(6 12)(7 14)(8 24)(9 19)(10 30)(11 17)(13 16)(18 23)(21 25)(22 32)(28 31)	S_{32}	x^3yx^3	29	(1 18 28 31 30 29 25 2 23 4 17 8 20 21 11 10 19 3 15 26 6 22 24 5 27 13 16 12 7)
2	18	(1 28)(2 9)(3 29)(4 7)(5 36)(6 24)(8 16)(10 12)(11 33)(13 21)(14 17)(15 35)(18 31)(19 25)(20 26)(22 32)(23 34)(27 30)	S_{36}	$xyx^2yx^2yx^2yx$	29	(1 15 26 17 35 21 14 4 16 18 7 22 25 20 28 34 13 23 12 5 24 29 10 9 31 32 6 33 36)
2	20	(1 39)(2 40)(3 10)(4 32)(5 35)(6 17)(7 14)(8 22)(9 19)(11 31)(12 38)(13 36)(15 26)(16 18)(20 28)(21 30)(23 33)(24 29)(25 27)(34 37)	S_{40}	$x^4xyxyxyx^4yx$	37	(1 25 27 23 12 8 40 39 18 21 3 37 36 9 19 11 14 32 13 17 24 4 6 30 22 20 38 26 34 31 16 33 35 29 15 28 2)

TABLE 2. Elements y such that $\text{Aut } \mathcal{D} \cong \{1\}$ for $b = 2$ ($x = (1 \ 2 \ \dots \ n)$)

To find elements y such that $\langle x, y \rangle = S_n$ for $x = (1 \ 2 \ \dots \ n)$, we make use of the following theorem.

Theorem 7.4. *Let $G \leq S_n$ be a primitive group which contains a cycle (p) of prime length p . Then either $G \geq A_n$ or $n \leq p + 2$.*

Proof. See [DM96, Theorem 3.3E]. □

By this theorem, for a permutation y of cycle type (2^q) , if $G = \langle x, y \rangle$ has no blocks with respect to any residue class of a divisor m of $n = 2q$ ($2 \leq m < n$), and if G contains an element w of cycle type (p_w) for a prime $p_w \leq n - 3$, then $G = S_n$ (since n is even, it cannot be A_n).

A list of such examples of y and w obtained by computation is shown in Table 2.

In practice, we generated random permutations y of cycle type (2^q) , produced several tens of thousands of elements of $\langle x, y \rangle$, and stopped as soon as an element w satisfying the required conditions was found; otherwise, we discarded y and tried again. By repeating this procedure, we completed the table.

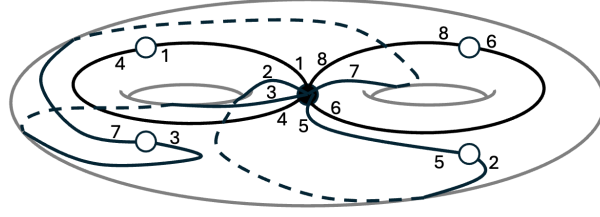
When $(b, q) = (2, 4)$, the monodromy group never becomes S_8 . Instead, it can be a group of order 336, in which case the automorphism group is trivial. A dessin realizing this case is shown in Figure 9.

7.4. The Case $b \geq 3$

Proposition 7.5. *Every passport $[n, b^q, n]$ of genus ≥ 2 with $b \geq 3$ and $q \geq 2$ admits a dessin with a trivial automorphism group.*

Proof. By (2.4), the genus is

$$\frac{n - (1 + q + 1)}{2} + 1 = \frac{n - q}{2} \geq 2.$$



$$x = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8), y = (1\ 4)(2\ 5)(3\ 7)(6\ 8) \\ |G| = 336, \text{Aut } \mathcal{D} \cong \{1\}$$

FIGURE 9. A dessin with trivial automorphism group having passport $[8, 2^4, 8]$

Therefore, we have $n - q \geq 4$ and $n \equiv q \pmod{2}$, and hence $n \geq 6$.

Our goal is to show that

$$\frac{I(b, q)}{T(b, q)} < \frac{N(b, q)}{T(b, q)}.$$

By Theorem 3.3,

$$\frac{N(b, q)}{T(b, q)} \geq \frac{2}{n+2} = \frac{2}{n(1 + \frac{2}{n})} \geq \frac{2}{n(1 + \frac{2}{6})} = \frac{3}{2n}.$$

Thus it suffices to prove that

$$\frac{I(b, q)}{T(b, q)} < \frac{3}{2n}.$$

Since there are $\tau(n) - 2$ integers m such that $m \mid n$ and $2 \leq m < n$ (where $\tau(n)$ denotes the divisor-counting function), it will be enough to show that for each such m ,

$$\frac{I_m}{T} < \frac{3}{2n\tau(n)}.$$

Since $\tau(n) \leq 2\sqrt{n}$, we have

$$\frac{3}{2n\tau(n)} < \frac{3}{4n^{\frac{3}{2}}} = \frac{3}{4(bq)^{\frac{3}{2}}}.$$

Thus it suffices to show

$$F(b, q, m) := \frac{I_m}{T} \cdot \frac{4}{3}(bq)^{\frac{3}{2}} < 1 \quad (7.14)$$

for all m such that $m \mid n$ and $2 \leq m < n$.

By (7.5),

$$F(b, q, m) \leq \frac{4}{3}m! b^{\frac{m}{2}+1} q^{\frac{3}{2}} m^{-q(b-1)} S(b, q, m), \quad (7.15)$$

where

$$S(b, q, m) = \sum_{\{(d_i, t_i)\}} \prod_i \frac{d_i^{\left(\frac{d_i q}{m} - 1\right)t_i}}{t_i!}. \quad (7.16)$$

Furthermore, from (7.6),

$$F(b, q, m) \leq \frac{4\sqrt{2\pi}}{3} b^{\frac{m}{2}+1} q^{\frac{3}{2}} m^{m-q(b-1)+\frac{1}{2}} e^{-m+\frac{1}{12m}} S(b, q, m).$$

Since $m \geq 2$, we have $e^{1/(12m)} \leq e^{1/24}$, and therefore

$$F(b, q, m) \leq C_0 e^{-m} b^{\frac{m}{2}+1} q^{\frac{3}{2}} m^{m-q(b-1)+\frac{1}{2}} S(b, q, m), \quad C_0 := \frac{4\sqrt{2\pi}}{3} e^{\frac{1}{24}}.$$

Taking logarithms gives

$$\begin{aligned} \log F(b, q, m) &\leq \log C_0 - m + \left(\frac{m}{2} + 1\right) \log b + \frac{3}{2} \log q + \left(m - q(b-1) + \frac{1}{2}\right) \log m \\ &\quad + \log S(b, q, m). \end{aligned} \quad (7.17)$$

What we want to show is that $\log F(b, q, m) < 0$.

For $S(b, q, m)$ in (7.16), let us define the set of admissible d_i for (3.9) by

$$D = \{d \in \mathbb{Z}_{>0} \mid d \leq m, d \mid b, m \mid dq\}. \quad (7.18)$$

Here,

$$\exp\left(\sum_{d \in D} d^{\frac{dq}{m}-1} x^d\right) = \prod_{d \in D} \exp\left(d^{\frac{dq}{m}-1} x^d\right) = \prod_{d \in D} \sum_{t=0}^{\infty} \frac{d^{\left(\frac{dq}{m}-1\right)t} x^{dt}}{t!}.$$

When we extract the coefficient of x^m on the right-hand side, the remaining terms are precisely the products and sums over all $\{(d_i, t_i)\}$ satisfying $\sum_i d_i t_i = m$. Since each $d_i \in D$, these satisfy (3.9). Thus, the expression coincides with (7.16). Therefore, by an estimate similar to that used in (7.11), we obtain

$$S(b, q, m) = [x^m] \exp\left(\sum_{d \in D} d^{\frac{dq}{m}-1} x^d\right), \quad (7.19)$$

and furthermore,

$$S(b, q, m) \leq \frac{\exp\left(\sum_{d \in D} d^{\frac{dq}{m}-1} R^d\right)}{R^m} \quad \text{for all } R > 0.$$

Define

$$g := \gcd(m, q), \quad d_0 := \frac{m}{g}, \quad q' := \frac{q}{g}, \quad b' := \frac{b}{d_0}.$$

Since $m \mid n (= bq)$, we have $d_0 \mid b$, and hence $b' \in \mathbb{Z}_{>0}$.

As $m \mid d_i q \Leftrightarrow d_0 \mid d_i$, the set of all d_i satisfying (3.9) coincides with the set of integers of the form

$$d_i = d_0 r, \quad 1 \leq r \leq g, \quad r \mid b'.$$

Thus, (7.18) can be written as

$$D = \{d_0 r \mid 1 \leq r \leq g, r \mid b'\}.$$

Focusing on the values of r , define

$$D_r = \{r \mid 1 \leq r \leq g, r \mid b'\}.$$

Then (7.19) becomes

$$\begin{aligned} S(b, q, m) &= [x^m] \exp\left(\sum_{r \in D_r} (d_0 r)^{\frac{q d_0 r}{m}-1} x^{d_0 r}\right) = [x^m] \exp\left(\sum_{r \in D_r} (d_0 r)^{\frac{q m r}{m g}-1} x^{d_0 r}\right) \\ &= [x^m] \exp\left(\sum_{r \in D_r} (d_0 r)^{q' r-1} x^{d_0 r}\right). \end{aligned} \quad (7.20)$$

Furthermore, for the parameters required in estimating $S(b, q, m)$, set

$$\alpha := d_0^{q'-1}, \quad \beta := \sum_{r \in D_r, r \geq 2} (d_0 r)^{q'r-1}. \quad (7.21)$$

In what follows, we proceed by considering the following cases according to the values of b , q , and m .

(A) $|D_r| = 1$ (i.e. $D_r = \{1\}$)

(A-1) $d_0 = b$

(A-2) $m < b$

(B) $|D_r| \geq 2$

(B-1) $\beta < g/2$

(B-2) $\beta \geq g/2$

(B-2-1) $m \mid q$

(B-2-2) $m \nmid q$

In each case, we proceed as follows:

- (1) Find an upper bound $G(b, q, m)$ satisfying $\log F(b, q, m) \leq G(b, q, m)$ and containing no factorials, summations, or products.
- (2) Show that $\partial^2 G / \partial m^2 > 0$. Thus, for each pair (b, q) , the function G is convex in m , and consequently its maximum occurs at one of the endpoints of the admissible range of m .
- (3) Determine the range of (b, q) for which G is negative at all relevant boundary points of m (namely, its minimum and maximum, and any junction points arising from case distinctions).

In this way, we show that $G < 0$ holds except for finitely many exceptional cases.

For the finitely many pairs (b, q) for which $G \geq 0$, we proceed as in Proposition 7.3 and explicitly find permutations x and y such that $\text{Aut } \mathcal{D} \cong \{1\}$.

(A) The case $|D_r| = 1$ (i.e. $D_r = \{1\}$).

In this case, there is no r satisfying both $2 \leq r \leq g$ and $r \mid b'$.

Moreover, since $d_0 \mid b$, we have $d_0 \leq b$.

If $d_0 = b$, then $b' = 1$, and hence $D_r = \{1\}$.

If $d_0 < b$, assume $m \geq b$. Then taking $r = b' = b/d_0 (> 1)$ yields $2 \leq r \leq m/d_0 = g$, contradicting the assumption.

Therefore, when $D_r = \{1\}$, we must have

$$d_0 = b \quad \text{or} \quad m < b.$$

Since $|D_r| = 1$, the only partition satisfying (3.9) is $\{(d_i, t_i)\} = \{(d_0, g)\}$. Hence,

$$S(b, q, m) = \frac{d_0^{\left(\frac{d_0 q}{m} - 1\right)g}}{g!} = \frac{d_0^{\left(\frac{mq}{gm} - 1\right)g}}{g!} = \frac{d_0^{q-g}}{g!}.$$

Substituting this into (7.15) and applying (7.4), we obtain

$$\begin{aligned} F(b, q, m) &\leq \frac{4}{3} m! b^{\frac{m}{2}+1} q^{\frac{3}{2}} m^{-q(b-1)} \frac{d_0^{q-g}}{g!} \\ &\leq \frac{\sqrt{2\pi m} \left(\frac{m}{e}\right)^m e^{\frac{1}{12m}}}{\sqrt{2\pi g} \left(\frac{g}{e}\right)^g e^{\frac{1}{12g+1}}} \cdot \frac{4}{3} b^{\frac{m}{2}+1} q^{\frac{3}{2}} m^{-q(b-1)} \left(\frac{m}{g}\right)^{q-g} \\ &= \frac{4}{3} b^{\frac{m}{2}+1} q^{\frac{3}{2}} m^{-q(b-1)} \left(\frac{m}{e}\right)^{m-g} \left(\frac{m}{g}\right)^{q+\frac{1}{2}} \exp\left(\frac{1}{12m} - \frac{1}{12g+1}\right). \end{aligned}$$

Since $m \geq 2$ and $m \geq g$, we have

$$\frac{1}{12m} - \frac{1}{12g+1} \leq \frac{1}{12m} - \frac{1}{12m+1} = \frac{1}{12m(12m+1)} \leq \frac{1}{12 \cdot 2(12 \cdot 2 + 1)} = \frac{1}{600}.$$

Therefore,

$$F(b, q, m) \leq C_1 b^{\frac{m}{2}+1} q^{\frac{3}{2}} m^{-q(b-1)} \left(\frac{m}{e}\right)^{m-g} \left(\frac{m}{g}\right)^{q+\frac{1}{2}}, \quad C_1 := \frac{4}{3} e^{\frac{1}{600}}.$$

Let $G(b, q, m)$ denote the logarithm of the right-hand side. Then

$$\begin{aligned} G(b, q, m) &= \log C_1 + \left(\frac{m}{2} + 1\right) \log b + (m - g)(\log m - 1) + \frac{1}{2}(\log m - \log g) \\ &\quad + \frac{3}{2} \log q - q((b - 2) \log m + \log g). \end{aligned} \quad (7.22)$$

(A-1) The subcase $d_0 = b$.

e.g. $(b, q, m) = (3, 2, 3), (5, 2, 5), (3, 4, 3), (3, 4, 6), (7, 2, 7)$

In this case, since $d_0 = m/g = b$, we have $g = m/b$ and $\log g = \log m - \log b$. Hence (7.22) becomes

$$\begin{aligned} G(b, q, m) &= \log C_1 + \left(\frac{m}{2} + 1\right) \log b + m \left(1 - \frac{1}{b}\right) (\log m - 1) + \frac{1}{2} \log b + \frac{3}{2} \log q \\ &\quad - q((b - 1) \log m - \log b), \end{aligned}$$

and

$$\frac{\partial^2 G}{\partial m^2} = \frac{b-1}{m} \left(\frac{q}{m} + \frac{1}{b}\right) \geq \frac{3-1}{m} \left(\frac{q}{m} + \frac{1}{b}\right) > 0.$$

Thus, for fixed b and q , the function G is convex in m . Since $g = m/b \geq 1$, we have $m \geq b$; moreover, from $m \mid n$ and $m < n$, it follows that $m \leq n/2 = bq/2$. Hence $b \leq m \leq bq/2$.

Setting $H_1(b, q) := G(b, q, b)$ and $H_2(b, q) := G(b, q, bq/2)$, we obtain

$$G(b, q, m) \leq \max\{H_1(b, q), H_2(b, q)\},$$

where

$$H_1(b, q) = \log C_1 - b + 1 + \frac{3}{2} \log q + \left(-(b-2)q + \frac{3}{2}b + \frac{1}{2}\right) \log b,$$

$$\frac{\partial H_1}{\partial q} = \frac{3}{2q} - (b-2) \log b \leq \frac{3}{2 \cdot 2} - (3-2) \log 3 < 0,$$

and

$$\begin{aligned} H_2(b, q) &= \log C_1 + \frac{q}{2}(b-1)(\log 2 - 1) + \frac{1}{4}(-bq + 6q + 6) \log b \\ &\quad + \frac{1}{2}(-q(b-1) + 3) \log q, \\ \frac{\partial H_2}{\partial q} &= \frac{3}{2q} + \frac{1}{2}(b-1)(\log 2 - \log q) - b + 1 + \left(-\frac{1}{4}b + \frac{3}{2}\right) \log b \\ &\leq \frac{3}{2 \cdot 2} + \frac{1}{2}(b-1)(\log 2 - \log 2) - b + 1 + \left(-\frac{1}{4}b + \frac{3}{2}\right) \log b \\ &= \frac{3}{4} - b + 1 + \left(-\frac{1}{4}b + \frac{3}{2}\right) \log b. \end{aligned}$$

Let the right-hand side be $g(b)$. Then

$$g'(b) = -\frac{5}{4} - \frac{1}{4} \log b + \frac{3}{2b} \leq -\frac{5}{4} - \frac{1}{4} \log 3 + \frac{3}{2 \cdot 3} < 0 \quad (b \geq 3).$$

Hence $g(b)$ is decreasing in b . Therefore

$$\frac{\partial H_2}{\partial q} \leq g(3) = \frac{3}{4} - 3 + 1 + \left(-\frac{1}{4} \cdot 3 + \frac{3}{2}\right) \log 3 < 0.$$

Hence both $H_1(b, q)$ and $H_2(b, q)$ are decreasing in q . Therefore, neither exceeds their value at $q = 2$.

Since $H_1(b, 2) = H_2(b, 2) = G(b, 2, b)$, we obtain

$$G(b, q, m) \leq G(b, 2, b) = \log C_1 + \frac{3}{2} \log 2 - b + 1 - \frac{b-9}{2} \log b.$$

Differentiating with respect to b , we have

$$\frac{d}{db} G(b, 2, b) = -\frac{3}{2} - \frac{\log b}{2} + \frac{9}{2b} \leq -\frac{3}{2} - \frac{\log 3}{2} + \frac{9}{2 \cdot 3} < 0 \quad (b \geq 3).$$

Thus $G(b, 2, b)$ is strictly decreasing for $b \geq 3$, and direct computation shows that it becomes negative for $b \geq 6$. Hence, in this range, we have $G(b, q, m) < 0$ for all q and m .

For $3 \leq b \leq 5$, taking into account that q must be even whenever b is even, direct computation shows that

$$\begin{aligned} &H_1(3, k) \ (2 \leq k \leq 5), H_1(4, 2), H_1(5, 2) > 0, \\ &H_1(3, 6), H_1(4, 4), H_1(5, 3) < 0, \\ &H_2(3, k) \ (2 \leq k \leq 4), H_2(4, 2), H_2(5, 2) > 0, \\ &H_2(3, 5), H_2(4, 4), H_2(5, 3) < 0. \end{aligned}$$

Since both H_1 and H_2 are decreasing in q , H_1 or H_2 becomes non-negative only for

$$(b, q) = (3, 2), (3, 3), (3, 4), (3, 5), (4, 2), (5, 2). \quad (7.23)$$

Therefore, when $|D_r| = 1$ and $d_0 = b$, inequality (7.14) holds except for these pairs.

(A-2) The subcase $m < b$.

e.g. $(b, q, m) = (3, 2, 2), (5, 2, 2), (3, 4, 2), (6, 2, 3), (6, 2, 4), (7, 2, 2)$

Differentiating (7.22) with respect to g gives

$$-(\log m - 1) - \frac{1}{2g} - \frac{q}{g} < -(\log 2 - 1) - 0 - 1 < 0,$$

therefore (7.22) is strictly decreasing in g .

Hence, letting $G_2(b, q, m)$ denote the expression obtained by substituting $g = 1$, we have

$$\begin{aligned} G(b, q, m) &\leq G_2(b, q, m) \\ &= \log C_1 + \left(\frac{m}{2} + 1\right) \log b + (m-1)(\log m - 1) + \frac{3}{2} \log q \\ &\quad - \left(q(b-2) - \frac{1}{2}\right) \log m. \end{aligned}$$

Moreover,

$$\frac{\partial^2 G_2}{\partial m^2} = \frac{1}{m} + \frac{q(b-2) + \frac{1}{2}}{m^2} \geq \frac{1}{m} + \frac{q(3-2) + \frac{1}{2}}{m^2} > 0.$$

Thus G_2 is convex in m . Since $2 \leq m < b$, its maximum is attained at the endpoints of this interval.

Setting

$$H_1(b, q) := G_2(b, q, 2), \quad H_2(b, q) := G_2(b, q, b),$$

we obtain

$$G(b, q, m) \leq G_2(b, q, m) \leq \max\{H_1(b, q), H_2(b, q)\},$$

where

$$\begin{aligned} H_1(b, q) &= \log C_1 + \left(\frac{3}{2} - q(b-2)\right) \log 2 - 1 + 2 \log b + \frac{3}{2} \log q, \\ \frac{\partial H_1}{\partial b} &= -q \log 2 + \frac{2}{b} \leq -2 \log 2 + \frac{2}{3} < 0, \\ \frac{\partial H_1}{\partial q} &= -(b-2) \log 2 + \frac{3}{2q} \begin{cases} = -(3-2) \log 2 + \frac{3}{2 \cdot 2} > 0 & (b=3, q=2) \\ \leq -(3-2) \log 2 + \frac{3}{2 \cdot 3} < 0 & (b=3, q \geq 3), \\ \leq -(4-2) \log 2 + \frac{3}{2 \cdot 2} < 0 & (b \geq 4) \end{cases} \end{aligned}$$

and

$$\begin{aligned} H_2(b, q) &= \log C_1 - b + 1 + \left(\frac{3b+1}{2} - q(b-2)\right) \log b + \frac{3}{2} \log q, \\ \frac{\partial H_2}{\partial b} &= \frac{1}{2} + \left(\frac{3}{2} - q\right) \log b - q \left(1 - \frac{2}{b}\right) + \frac{1}{2b} \\ &\leq \frac{1}{2} + \left(\frac{3}{2} - 2\right) \log 3 - 2 \left(1 - \frac{2}{3}\right) + \frac{1}{2 \cdot 3} < 0, \\ \frac{\partial H_2}{\partial q} &= -(b-2) \log b + \frac{3}{2q} \leq -(3-2) \log 3 + \frac{3}{2 \cdot 2} < 0. \end{aligned}$$

Thus H_1 is strictly decreasing in b , and, except for the case $(b, q) = (3, 2)$, also strictly decreasing in q . Similarly, H_2 is strictly decreasing in both b and q . Direct computation shows that

$$\begin{aligned} &H_1(3, k) \ (2 \leq k \leq 8), H_1(4, 2), H_1(5, 2) > 0, \\ &H_1(3, 9), H_1(4, 4), H_1(5, 3), H_1(6, 2) < 0, \\ &H_2(3, k) \ (2 \leq k \leq 5), H_2(4, 2), H_2(5, 2) > 0, \\ &H_2(3, 6), H_2(4, 4), H_2(5, 3), H_2(6, 2) < 0. \end{aligned}$$

Therefore, when $|D_r| = 1$ and $m < b$, inequality (7.14) holds except in the following cases:

$$(b, q) = (3, k) \ (2 \leq k \leq 8), (4, 2), (5, 2). \quad (7.24)$$

(B) The case $|D_r| \geq 2$.

In this case, since there exists an integer r such that $2 \leq r \leq g$ and $r \mid b'$, we must have $g \geq 2$ and $b' \geq 2$.

Moreover, when $(b, q) = (3, 2)$, we have $n = 6$, and hence $m = 2$ or 3 . In this case,

$$\begin{aligned} m = 2 &\Rightarrow g = 2, \ b' = 3 \Rightarrow D_r = \{1\}, \\ m = 3 &\Rightarrow g = 1 \Rightarrow D_r = \{1\}. \end{aligned}$$

Thus, for $(b, q) = (3, 2)$, we always have $D_r = \{1\}$. Therefore, this pair can be excluded from the present case.

From (7.20) we have

$$\begin{aligned}
S(b, q, m) &= [x^m] \exp \left(\sum_{r \in D_r} (d_0 r)^{q' r - 1} x^{d_0 r} \right) \\
&= [y^g] \exp \left(\sum_{r \in D_r} (d_0 r)^{q' r - 1} y^r \right) \quad (y = x^{d_0}) \\
&\leq \frac{\exp \left(\sum_{r \in D_r} (d_0 r)^{q' r - 1} R^r \right)}{R^g} \quad (\text{for all } R > 0).
\end{aligned}$$

Therefore,

$$\log S(b, q, m) \leq \sum_{r \in D_r} (d_0 r)^{q' r - 1} R^r - g \log R \quad (\text{for all } R > 0). \quad (7.25)$$

Using α and β defined in (7.21), we obtain, for $0 < R \leq 1$,

$$\log S(b, q, m) \leq \alpha R + \beta R^2 - g \log R.$$

Differentiating the right-hand side with respect to R yields $(2\beta R^2 + \alpha R - g)/R$, and hence it attains its minimum at $R = (-\alpha + \sqrt{\alpha^2 + 8\beta g})/(4\beta)$. Taking a nearby point $R = \sqrt{8\beta g}/(4\beta) = \sqrt{g/(2\beta)}$ and setting

$$R_0 := \min \left\{ \sqrt{\frac{g}{2\beta}}, 1 \right\}, \quad (7.26)$$

we have $0 < R_0 \leq 1$, and hence

$$\log S(b, q, m) \leq \alpha R_0 + \beta R_0^2 - g \log R_0. \quad (7.27)$$

(B-1) The subcase $\beta < g/2$.

e.g. $(b, q, m) = (4, 10, 20), (4, 12, 24), (4, 14, 28), (3, 19, 19), (3, 20, 20)$

In this situation, we have $R_0 = 1$ by (7.26). Since the sum β contains at least one term with $r \geq 2$, we obtain

$$\beta = \sum_{r \in D_r, r \geq 2} (d_0 r)^{q' r - 1} \geq 2^{2-1} = 2,$$

and since $\beta < g/2$, we obtain $g \geq 5$, which implies $q, m \geq 5$.

Assume $g = 5$. Then:

- If q is odd, then b is also odd, and therefore every divisor $r \geq 2$ of b' is at least 3. Hence $\beta \geq 3^{3-1} = 9$, but since $\beta < g/2$, this would force $g \geq 19$, a contradiction.
- If q is even, then $q \geq 10$ since $5 \mid q$, and we have $q' = q/5 \geq 2$.

Therefore, $\beta \geq 2^{2q'-1} \geq 2^3 = 8 > 5/2 = g/2$, again a contradiction.

Thus we must have $g \geq 6$ (and in fact $g \geq 19$ when q is odd). Consequently, $q, m \geq 6$.

We have

$$\frac{\alpha}{\beta} \leq \frac{d_0^{q'-1}}{(2d_0)^{2q'-1}} = \frac{1}{2^{2q'-1} d_0^{q'}} \leq \frac{1}{2},$$

and $R_0 = 1$. Therefore, from (7.27) we obtain

$$\log S(b, q, m) \leq \alpha + \beta \leq \frac{3}{2}\beta < \frac{3}{4}g \leq \frac{3}{4}m.$$

Substituting this into the right-hand side of (7.17), we obtain the following upper bound, which we denote by $G(b, q, m)$:

$$G(b, q, m) = \log C_0 - \frac{1}{4}m + \left(\frac{m}{2} + 1 \right) \log b + \frac{3}{2} \log q + \left(m - q(b-1) + \frac{1}{2} \right) \log m,$$

and

$$\frac{\partial^2 G}{\partial m^2} = \frac{1}{m} + \frac{q(b-1) - \frac{1}{2}}{m^2} \geq \frac{1}{m} + \frac{6(3-1) - \frac{1}{2}}{m^2} > 0.$$

Thus G is convex in m . Since $6 \leq m \leq bq/2$, the maximum of G over this range is attained at one of the two endpoints.

Let $H_1(b, q) := G(b, q, 6)$, $H_2(b, q) := G(b, q, bq/2)$, so that

$$G(b, q, m) \leq \max\{H_1(b, q), H_2(b, q)\},$$

where

$$H_1(b, q) = \left(-q(b-1) + \frac{13}{2}\right) \log 6 - \frac{3}{2} + \log C_0 + 4 \log b + \frac{3}{2} \log q,$$

$$\frac{\partial H_1}{\partial b} = -q \log 6 + \frac{4}{b} \leq -6 \log 6 + \frac{4}{3} < 0,$$

$$\frac{\partial H_1}{\partial q} = (1-b) \log 6 + \frac{3}{2q} \leq (1-3) \log 6 + \frac{3}{2 \cdot 6} < 0,$$

and

$$H_2(b, q) = \left(-\frac{bq}{4} + q + \frac{3}{2}\right) \log b + \left(-\frac{bq}{2} + q + 2\right) \log q + \left(\frac{bq}{2} - q - \frac{1}{2}\right) \log 2 \\ + \log C_0 - \frac{bq}{8},$$

$$\frac{\partial H_2}{\partial b} = \frac{q}{8} \left(-2 \log b - 4 \log q - 3 + 4 \log 2 + \frac{8}{b}\right) + \frac{3}{2b} \\ \leq \frac{6}{8} \left(-2 \log 3 - 4 \log 6 - 3 + 4 \log 2 + \frac{8}{3}\right) + \frac{3}{2 \cdot 3} < 0,$$

$$\frac{\partial H_2}{\partial q} = \left(1 - \frac{b}{4}\right) \log b + \left(1 - \frac{1}{2}b\right) \log q + b \left(\frac{1}{2} \log 2 - \frac{5}{8}\right) - \log 2 + 1 + \frac{2}{q} \\ \leq \left(1 - \frac{3}{4}\right) \log 3 + \left(1 - \frac{1}{2} \cdot 3\right) \log 6 + 3 \left(\frac{1}{2} \log 2 - \frac{5}{8}\right) - \log 2 + 1 + \frac{2}{6} < 0.$$

Thus both H_1 and H_2 are strictly decreasing in b and q . Direct computation shows that

$$H_1(3, 6) < 0, \\ H_2(3, 6), H_2(3, 7), H_2(3, 8) > 0, \\ H_2(3, 9), H_2(4, 6) < 0.$$

Therefore, in the case $|D_r| \geq 2$ and $\beta < g/2$, inequality (7.14) holds except for the following pairs:

$$(b, q) = (3, 6), (3, 7), (3, 8). \quad (7.28)$$

(B-2) The Subcase $\beta \geq g/2$.

In this situation, we have $R_0 = \sqrt{g/(2\beta)}$ by (7.26). Since D_r contains at least two elements not exceeding g , we have $g \geq 2$, and since b' has at least two divisors, we also have $b' \geq 2$.

Let p_0 be the smallest prime divisor of $b' = b/d_0 = bg/m$ and write the prime factorization of b' as

$$b' = p_0^{e_0} \cdots p_{k-1}^{e_{k-1}} \quad (k \geq 1, p_0 < \cdots < p_{k-1}, e_i \geq 1).$$

Then

$$D_r = \{r \mid 1 \leq r \leq g, r \mid b'\} = \{r = p_0^{f_0} \cdots p_{k-1}^{f_{k-1}} \mid 0 \leq f_i \leq e_i, r \leq g\}.$$

Fixing $f_1, \dots, f_{k-1}$, consider the p_0 -adic chain

$$l, p_0 l, p_0^2 l, \dots, p_0^{e_0} l \quad (l = p_1^{f_1} \cdots p_{k-1}^{f_{k-1}}), \quad (7.29)$$

then every $r \in D_r$ lies in exactly one such chain.

For β defined by (7.21), the ratio of two adjacent terms within the same chain corresponding to $r = p_0^{f_0} l$ is

$$\begin{aligned} \frac{(d_0 p_0^{f_0+1} l)^{q' p_0^{f_0+1} l-1}}{(d_0 p_0^{f_0} l)^{q' p_0^{f_0} l-1}} &= \left(\frac{d_0 p_0^{f_0+1} l}{d_0 p_0^{f_0} l} \right)^{q' p_0^{f_0} l-1} (d_0 p_0^{f_0+1} l)^{q' (p_0-1) p_0^{f_0} l} \\ &= p_0^{q' p_0^{f_0} l-1} (d_0 p_0^{f_0+1} l)^{q' (p_0-1) p_0^{f_0} l} \\ &\geq p_0^{q'-1} (d_0 p_0)^{q' (p_0-1)} = d_0^{q' (p_0-1)} p_0^{q' p_0-1} \\ &\geq 2^{2q'-1} d_0^{q'}. \end{aligned}$$

Let this final expression be denoted by Λ . If M_l is the largest value of $(d_0 r)^{q' r-1}$ within the chain corresponding to l , then the total contribution β_l of that chain satisfies

$$\beta_l < M_l \frac{\Lambda}{\Lambda - 1}.$$

Let $\tau(b')$ denote the number of divisors of b' . Then we have

$$\tau(b') = (e_0 + 1) \cdots (e_{k-1} + 1).$$

Moreover, it is well known that $\tau(b') \leq 2\sqrt{b'}$.

The number c of the chains (7.29) is equal to the number of divisors of b' that are not divisible by p_0 . Hence

$$c = \frac{\tau(b')}{e_0 + 1} \leq \frac{\tau(b')}{2} \leq \sqrt{b'}.$$

Therefore, if M denotes the largest value among the M_l , then

$$\beta \leq \sum_l \beta_l < \frac{\Lambda}{\Lambda - 1} \sum_l M_l \leq M \sqrt{b'} \frac{\Lambda}{\Lambda - 1}.$$

Since $M = (d_0 r)^{q' r-1}$ for some $r \in D_r$, and since every such r satisfies $r \leq \min\{g, b'\} = \min\{g, bg/m\} = g \min\{1, b/m\}$, we obtain

$$M \leq \begin{cases} (d_0 g)^{q' g-1} = \left(\frac{m}{g} \cdot g \right)^{\frac{q}{g} \cdot g-1} = m^{q-1} & (m \leq b) \\ (d_0 b')^{q' \cdot \frac{bg}{m}-1} = \left(d_0 \cdot \frac{b}{d_0} \right)^{\frac{q}{g} \cdot \frac{bg}{m}-1} = b^{\frac{bg}{m}-1} & (m \geq b) \end{cases}.$$

Thus,

$$\beta < \begin{cases} \frac{\Lambda}{\Lambda - 1} \sqrt{b'} m^{q-1} = \frac{\Lambda}{\Lambda - 1} \sqrt{\frac{bg}{m}} m^{q-1} & (m \leq b) \\ \frac{\Lambda}{\Lambda - 1} \sqrt{b'} b^{\frac{bg}{m}-1} = \frac{\Lambda}{\Lambda - 1} \sqrt{\frac{bg}{m}} b^{\frac{bg}{m}-1} & (m \geq b) \end{cases}, \quad (7.30)$$

and in (7.25), for $0 < R \leq 1$,

$$\log S(b, q, m) \leq \sum_{r \in D_r} (d_0 r)^{q' r-1} R^r - g \log R$$

$$\leq \alpha R + \beta R^2 - g \log R,$$

where α and β are defined by (7.21).

We can put $R = R_0 = \sqrt{g/(2\beta)}$, and therefore

$$\begin{aligned} \log S(b, q, m) &\leq \alpha R_0 + \beta R_0^2 - g \log R_0 \\ &= \alpha \sqrt{\frac{g}{2\beta}} + \frac{g}{2} \left(1 + \log \frac{2\beta}{g} \right) \\ &= \alpha \sqrt{\frac{g}{2\beta}} + \frac{g}{2} (1 + \log 2 + \log \beta - \log g). \end{aligned} \quad (7.31)$$

(B-2-1) The subcase $m \mid q$.

e.g. $(b, q, m) = (4, 2, 2), (3, 3, 3), (3, 4, 4), (6, 2, 2), (3, 5, 5)$

In this case, we have $2 \leq g = m \leq q$, together with $d_0 = m/g = 1$ and $b' = b/d_0 = b$. Then

$$\begin{aligned} \Lambda &= 2^{2q'-1} d_0^{q'} = 2^{2q'-1} \geq 2, \\ \frac{\Lambda}{\Lambda - 1} &\leq \frac{2}{2 - 1} = 2, \\ \alpha \sqrt{\frac{g}{2\beta}} &= d_0^{q'-1} \sqrt{\frac{g}{2\beta}} = \sqrt{\frac{g}{2\beta}} \leq 1, \end{aligned}$$

and by (7.30),

$$\beta < \begin{cases} 2\sqrt{b} m^{q-1} & (m \leq b) \\ 2\sqrt{b} b^{\frac{bq}{m}-1} & (m \geq b) \end{cases}.$$

Therefore, by (7.31), for $m \leq b$ we have

$$\begin{aligned} \log S(b, q, m) &\leq 1 + \frac{g}{2} \left(1 + \log 2 + \log 2\sqrt{b} m^{q-1} - \log g \right) \\ &= 1 + \frac{m}{2} \left(1 + \log 2 + \log 2\sqrt{b} m^{q-1} - \log m \right) \\ &= 1 + \frac{m}{2} \left(1 + 2 \log 2 + \frac{1}{2} \log b + (q-2) \log m \right), \end{aligned}$$

and for $m \geq b$ we have

$$\begin{aligned} \log S(b, q, m) &\leq 1 + \frac{g}{2} \left(1 + \log 2 + \log 2\sqrt{b} b^{\frac{bq}{m}-1} - \log g \right) \\ &= 1 + \frac{m}{2} \left(1 + \log 2 + \log 2\sqrt{b} b^{\frac{bq}{m}-1} - \log m \right) \\ &= 1 + \frac{m}{2} \left(1 + 2 \log 2 + \left(\frac{bq}{m} - \frac{1}{2} \right) \log b - \log m \right). \end{aligned}$$

Substituting these estimates into the right-hand side of (7.17) and denoting the resulting quantity by $G(b, q, m)$, we obtain

$$\begin{aligned} G(b, q, m) &= \log C_0 - m + \left(\frac{m}{2} + 1 \right) \log b + \frac{3}{2} \log q + \left(m - q(b-1) + \frac{1}{2} \right) \log m \\ &\quad + \begin{cases} 1 + \frac{m}{2} \left(1 + 2 \log 2 + \frac{1}{2} \log b + (q-2) \log m \right) & (m \leq b) \\ 1 + \frac{m}{2} \left(1 + 2 \log 2 + \left(\frac{bq}{m} - \frac{1}{2} \right) \log b - \log m \right) & (m \geq b) \end{cases}, \end{aligned}$$

and

$$\frac{\partial^2 G}{\partial m^2} = \begin{cases} \frac{q(2b+m-2)-1}{2m^2} \geq \frac{2(2 \cdot 3 + 2 - 2) - 1}{2m^2} > 0 & (m \leq b) \\ \frac{2q(b-1)+m-1}{2m^2} \geq \frac{2 \cdot 2(3-1) + 2 - 1}{2m^2} > 0 & (m \geq b) \end{cases}.$$

Therefore, in both ranges $m \leq b$ and $m \geq b$, G is convex in m . Since $2 \leq m \leq q$, we need only evaluate G at $m = 2$ and $m = q$ when $q < b$, and at $m = 2$, $m = b$, and $m = q$ when $q \geq b$.

Let $H_1(b, q) := G(b, q, 2)$, $H_2(b, q) := G(b, q, q)$, and $H_3(b, q) := G(b, q, b)$. Then

$$G(b, q, m) \leq \begin{cases} \max\{H_1(b, q), H_2(b, q)\} & (q < b) \\ \max\{H_1(b, q), H_2(b, q), H_3(b, q)\} & (q \geq b) \end{cases}.$$

Since $m = 2$ always satisfies $m \leq b$, we have

$$\begin{aligned} H_1(b, q) &= \log C_0 + \left(-bq + 2q + \frac{5}{2}\right) \log 2 + \frac{5}{2} \log b + \frac{3}{2} \log q, \\ \frac{\partial H_1}{\partial b} &= -q \log 2 + \frac{5}{2b} \leq -2 \log 2 + \frac{5}{2 \cdot 3} < 0, \\ \frac{\partial H_1}{\partial q} &= -(b-2) \log 2 + \frac{3}{2q} \leq \begin{cases} -(3-2) \log 2 + \frac{3}{2 \cdot 3} < 0 & (b = 3) \\ -(4-2) \log 2 + \frac{3}{2 \cdot 2} < 0 & (b \geq 4) \end{cases}. \end{aligned}$$

The last inequality holds because $(b, q) \neq (3, 2)$.

When $q < b$, note that $(b, q) \neq (3, 2)$ implies $b \geq 4$, and when $b = 4$ the integer q must be even, hence $q = 2$. Under these assumptions we have

$$\begin{aligned} H_2(b, q) &= \log C_0 + 1 + \left(\log 2 - \frac{1}{2}\right)q + \left(\frac{3}{4}q + 1\right) \log b + \left(-bq + \frac{1}{2}q^2 + q + 2\right) \log q, \\ \frac{\partial H_2}{\partial b} &= q \left(\frac{3}{4b} - \log q\right) + \frac{1}{b} \leq 2 \left(\frac{3}{4 \cdot 3} - \log 2\right) + \frac{1}{3} < 0, \\ \frac{\partial H_2}{\partial q} &= (-b + q + 1) \log q - b + \frac{q}{2} + \frac{3}{4} \log b + \frac{1}{2} + \log 2 + \frac{2}{q} \\ &\quad \begin{cases} = (-4 + 2 + 1) \log 2 - 4 + \frac{2}{2} + \frac{3}{4} \log 4 + \frac{1}{2} + \log 2 + \frac{2}{2} < 0 & (b = 4, q = 2) \\ = 0 \log q - b + \frac{b-1}{2} + \frac{3}{4} \log b + \frac{1}{2} + \log 2 + \frac{2}{b-1} < 0 & (b \geq 5, q = b-1) \\ \leq -\log 2 - b + \frac{b-2}{2} + \frac{3}{4} \log b + \frac{1}{2} + \log 2 + \frac{2}{2} < 0 & (b \geq 5, q \leq b-2) \end{cases} \end{aligned}$$

When $q \geq b$, we have

$$\begin{aligned} H_2(b, q) &= \log C_0 + 1 + \left(\log 2 - \frac{1}{2}\right)q + \left(\frac{bq}{2} + \frac{1}{4}q + 1\right) \log b + \left(-bq + \frac{3}{2}q + 2\right) \log q, \\ \frac{\partial H_2}{\partial b} &= \frac{q}{2} \left(\log b - 2 \log q + 1 + \frac{1}{2b}\right) + \frac{1}{b} \end{aligned}$$

$$\begin{cases}
= \frac{3}{2} \left(\log 3 - 2 \log 3 + 1 + \frac{1}{2 \cdot 3} \right) + \frac{1}{3} > 0 & (b = q = 3) \\
= \frac{b}{2} \left(\log b - 2 \log b + 1 + \frac{1}{2b} \right) + \frac{1}{b} < 0 & (4 \leq b = q), \\
\leq \frac{b+1}{2} \left(\log b - 2 \log(b+1) + 1 + \frac{1}{2b} \right) + \frac{1}{b} < 0 & (3 \leq b < q)
\end{cases}$$

$$\begin{aligned}
\frac{\partial H_2}{\partial q} &= \left(\frac{b}{2} + \frac{1}{4} \right) \log b + \left(-b + \frac{3}{2} \right) \log q - b + 1 + \log 2 + \frac{2}{q} \\
&\leq \left(\frac{b}{2} + \frac{1}{4} \right) \log b + \left(-b + \frac{3}{2} \right) \log b - b + 1 + \log 2 + \frac{2}{b} \\
&= \left(-\frac{b}{2} + \frac{7}{4} \right) \log b - b + 1 + \log 2 + \frac{2}{b} \\
&\begin{cases}
= \left(-\frac{3}{2} + \frac{7}{4} \right) \log 3 - 3 + 1 + \log 2 + \frac{2}{3} < 0 & (b = 3) \\
\leq \underbrace{\left(-\frac{b}{2} + \frac{7}{4} \right) \log b - 4 + 1 + \log 2 + \frac{2}{4}}_{< 0} < 0 & (b \geq 4),
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
H_3(b, q) &= \log C_0 + 1 + \left(\log 2 - \frac{1}{2} \right) b + \left(-\frac{bq}{2} + \frac{3}{4}b + q + \frac{3}{2} \right) \log b + \frac{3}{2} \log q, \\
\frac{\partial H_3}{\partial b} &= \left(\frac{3}{4} - \frac{1}{2}q \right) \log b + q \left(\frac{1}{b} - \frac{1}{2} \right) + \frac{1}{4} + \log 2 + \frac{3}{2b} \\
&\begin{cases}
= \left(\frac{3}{4} - \frac{1}{2} \cdot 3 \right) \log 3 + 3 \left(\frac{1}{3} - \frac{1}{2} \right) + \frac{1}{4} + \log 2 + \frac{3}{2 \cdot 3} > 0 & (b = q = 3) \\
\leq \left(\frac{3}{4} - \frac{1}{2} \cdot 4 \right) \log 3 + 4 \left(\frac{1}{3} - \frac{1}{2} \right) + \frac{1}{4} + \log 2 + \frac{3}{2 \cdot 3} < 0 & (b = 3, q > b), \\
\leq \left(\frac{3}{4} - \frac{1}{2} \cdot 4 \right) \log 4 + 4 \left(\frac{1}{4} - \frac{1}{2} \right) + \frac{1}{4} + \log 2 + \frac{3}{2 \cdot 4} < 0 & (b \geq 4)
\end{cases} \\
\frac{\partial H_3}{\partial q} &= \left(1 - \frac{b}{2} \right) \log b + \frac{3}{2q} \leq \left(1 - \frac{3}{2} \right) \log 3 + \frac{3}{2 \cdot 3} < 0.
\end{aligned}$$

Therefore, except for H_2 and H_3 in the case $b = q = 3$, all of H_1 , H_2 , and H_3 are decreasing in both b and q .

Taking into account that q must be even whenever b is even, direct computation gives

$$\begin{aligned}
&H_1(3, k) \ (3 \leq k \leq 13), H_1(4, 2), H_1(4, 4), H_1(4, 6), \\
&H_1(5, 2), H_1(5, 3), H_1(5, 4), H_1(6, 2), H_1(7, 2), H_1(8, 2) > 0, \\
&H_1(3, 14), H_1(4, 8), H_1(5, 5), H_1(6, 4), H_1(7, 3), H_1(9, 2) < 0, \\
&H_2(3, k) \ (3 \leq k \leq 7), H_2(4, 2), H_2(4, 4), H_2(4, 6), \\
&H_2(5, 2), H_2(5, 3), H_2(5, 4), H_2(5, 5), H_2(6, 2), H_2(7, 2), H_2(8, 2) > 0, \\
&H_2(3, 8), H_2(4, 8), H_2(5, 6), H_2(6, 4), H_2(7, 3), H_2(9, 2) < 0, \\
&H_3(3, k) \ (3 \leq k \leq 20), H_3(4, 4), H_3(4, 6), H_3(4, 8), H_3(5, 5) > 0, \\
&H_3(3, 21), H_3(4, 10), H_3(5, 6) < 0.
\end{aligned}$$

Therefore, in the case $|D_r| \geq 2$, $\beta \geq g/2$, and $m \mid q$, the inequality (7.14) holds except for the following pairs:

$$(b, q) = (3, k) \ (3 \leq k \leq 20), \ (4, k) \ (k = 2, 4, 6, 8), \ (5, k) \ (2 \leq k \leq 5), \\ (6, 2), (7, 2), (8, 2). \quad (7.32)$$

(B-2-2) The subcase $m \nmid q$.

e.g. $(b, q, m) = (4, 2, 4), (6, 2, 6), (4, 4, 8), (8, 2, 4), (8, 2, 8)$

In this case, since $|D_r| \geq 2$, there exists an integer $r \in D_r$ with $2 \leq r \leq g$ and $r \mid b'$.

Moreover, since $m \nmid q$, we have $g = \gcd(m, q) < m$, and since $g \mid m$, this implies $g \leq m/2$. Therefore $2 \leq g \leq m/2$, $m \geq 4$, and $d_0 = m/g \geq 2$. Furthermore, since $b = d_0 b' \geq 2b'$ and $b' \geq 2$, we obtain $b \geq 4$.

Since D_r contains such an $r \geq 2$, the sum in (7.21) contains at least one term with $r \geq 2$, and hence

$$\beta = \sum_{r \in D_r, r \geq 2} (d_0 r)^{q'r-1} \geq (2d_0)^{2q'-1}.$$

It follows that, in (7.31),

$$\alpha \sqrt{\frac{g}{2\beta}} = d_0^{q'-1} \sqrt{\frac{g}{2\beta}} \leq d_0^{q'-1} \sqrt{\frac{g}{2}} (2d_0)^{-q'+\frac{1}{2}} = 2^{-q'+\frac{1}{2}} d_0^{-\frac{1}{2}} \sqrt{\frac{g}{2}} \\ \leq (2d_0)^{-\frac{1}{2}} \sqrt{\frac{g}{2}} = \sqrt{\frac{g}{2m}} \sqrt{\frac{g}{2}} = \frac{g}{2\sqrt{m}} \leq \frac{\frac{m}{2}}{2\sqrt{m}} = \frac{\sqrt{m}}{4}. \quad (7.33)$$

Since

$$\Lambda = 2^{2q'-1} d_0^{q'} \geq 2^{2 \cdot 1-1} \cdot 2^1 = 4,$$

by (7.30), we obtain

$$\beta < \begin{cases} \frac{4}{3} \sqrt{\frac{bg}{m}} m^{q-1} & (m \leq b) \\ \frac{4}{3} \sqrt{\frac{bg}{m}} b^{\frac{bq}{m}-1} & (m \geq b) \end{cases}. \quad (7.34)$$

(i) The subcase $m \leq b$.

By (7.31), (7.33), and (7.34), we obtain the following estimate for $m \leq b$:

$$\log S(b, q, m) \leq \frac{\sqrt{m}}{4} + \frac{g}{2} \left(1 + \log 2 + \log \frac{4}{3} \sqrt{\frac{bg}{m}} m^{q-1} - \log g \right) \\ = \frac{\sqrt{m}}{4} + \frac{g}{2} \left(1 + 3 \log 2 - \log 3 + \frac{1}{2} (\log b - \log g) + \left(q - \frac{3}{2} \right) \log m \right). \quad (7.35)$$

Let the expression in parentheses be denoted by $f_1(b, q, m)$. Since $2g \leq m \leq b$, it follows that

$$f_1(b, q, m) = 1 + 3 \log 2 - \log 3 + \frac{1}{2} \log \frac{b}{g} + \left(q - \frac{3}{2} \right) \log m \\ \geq 1 + 3 \log 2 - \log 3 + \frac{1}{2} \log 2 + \left(2 - \frac{3}{2} \right) \log 2 > 0.$$

Therefore, applying the bound $2 \leq g \leq m/2$ to (7.35), we obtain

$$\begin{aligned}\log S(b, q, m) &\leq \frac{\sqrt{m}}{4} + \frac{m}{4} \left(1 + 3 \log 2 - \log 3 + \frac{1}{2} (\log b - \log 2) + \left(q - \frac{3}{2} \right) \log m \right) \\ &= \frac{\sqrt{m}}{4} + \frac{m}{4} \left(1 + \frac{5}{2} \log 2 - \log 3 + \frac{1}{2} \log b + \left(q - \frac{3}{2} \right) \log m \right).\end{aligned}$$

Substituting this into the right-hand side of (7.17) and denoting the resulting expression by $G(b, q, m)$, we obtain

$$\begin{aligned}G(b, q, m) &= \log C_0 - m + \left(\frac{m}{2} + 1 \right) \log b + \frac{3}{2} \log q + \left(m - q(b-1) + \frac{1}{2} \right) \log m \\ &\quad + \frac{\sqrt{m}}{4} + \frac{m}{4} \left(1 + \frac{5}{2} \log 2 - \log 3 + \frac{1}{2} \log b + \left(q - \frac{3}{2} \right) \log m \right),\end{aligned}\quad (7.36)$$

and

$$\begin{aligned}\frac{\partial^2 G}{\partial m^2} &= \frac{2q+5}{8m} + \frac{q(b-1) - \frac{1}{2} - \frac{1}{16}\sqrt{m}}{m^2} \geq \frac{2(b-1) - \frac{1}{2} - \frac{1}{16}\sqrt{b}}{m^2} \\ &= \frac{\sqrt{b}(2\sqrt{b} - \frac{1}{16}) - \frac{5}{2}}{m^2} \geq \frac{\sqrt{4}(2\sqrt{4} - \frac{1}{16}) - \frac{5}{2}}{m^2} > 0.\end{aligned}$$

Thus G is convex in m . Since $4 \leq m \leq b$, the maximum is attained at one of the boundary points.

Setting $H_1(b, q) := G(b, q, 4)$ and $H_2(b, q) = G(b, q, b)$, we obtain

$$G(b, q, m) \leq \max\{H_1(b, q), H_2(b, q)\}.$$

Substituting $m = 4$ into (7.36), we obtain

$$\begin{aligned}H_1(b, q) &= \log C_0 + \left(-2bq + 4q + \frac{17}{2} \right) \log 2 - \log 3 - \frac{5}{2} + \frac{7}{2} \log b + \frac{3}{2} \log q, \\ \frac{\partial H_1}{\partial b} &= -2q \log 2 + \frac{7}{2b} \leq -2 \cdot 2 \log 2 + \frac{7}{2 \cdot 4} < 0, \\ \frac{\partial H_1}{\partial q} &= (-2b + 4) \log 2 + \frac{3}{2q} \leq (-2 \cdot 4 + 4) \log 2 + \frac{3}{2 \cdot 2} < 0.\end{aligned}$$

Substituting $m = b$ into (7.36), we obtain

$$\begin{aligned}H_2(b, q) &= \log C_0 + \frac{\sqrt{b}}{4} - \frac{3}{4}b + \frac{5}{8}b \log 2 - \frac{b}{4} \log 3 + \left(-\frac{3}{4}bq + \frac{5}{4}b + q + \frac{3}{2} \right) \log b \\ &\quad + \frac{3}{2} \log q, \\ \frac{\partial H_2}{\partial b} &= \frac{-3q+5}{4} \log b + \left(-\frac{3}{4} + \frac{1}{b} \right) q + \frac{5}{8} \log 2 - \frac{1}{4} \log 3 + \frac{1}{2} + \frac{3}{2b} + \frac{1}{8\sqrt{b}} \\ &\leq \frac{-3 \cdot 2 + 5}{4} \log 4 + \left(-\frac{3}{4} + \frac{1}{4} \right) \cdot 2 + \frac{5}{8} \log 2 - \frac{1}{4} \log 3 + \frac{1}{2} + \frac{3}{2 \cdot 4} + \frac{1}{8\sqrt{4}} < 0, \\ \frac{\partial H_2}{\partial q} &= \left(1 - \frac{3}{4}b \right) \log b + \frac{3}{2q} \leq \left(1 - \frac{3}{4} \cdot 4 \right) \log 4 + \frac{3}{2 \cdot 2} < 0.\end{aligned}$$

Thus, both H_1 and H_2 are decreasing with respect to both b and q .

Noting that q must be even whenever b is even, we compute the values as follows:

$$\begin{aligned}H_1(4, 2), H_1(5, 2) &> 0, \\ H_1(4, 4), H_1(5, 3), H_1(6, 2) &< 0, \\ H_2(k, 2) \ (4 \leq k \leq 9) &> 0,\end{aligned}$$

$$H_2(4, 4), H_2(5, 3), H_2(10, 2) < 0.$$

Therefore, we have $G(b, q, m) < 0$ for all admissible (b, q, m) , except when

$$(b, q) = (k, 2) \quad (4 \leq k \leq 9).$$

(ii) The subcase $m \geq b$.

For $m \geq b$, we have the bound

$$\begin{aligned} \log S(b, q, m) &\leq \frac{\sqrt{m}}{4} + \frac{g}{2} \left(1 + \log 2 + \log \frac{4}{3} \sqrt{\frac{bg}{m}} b^{\frac{bg}{m}-1} - \log g \right) \\ &= \frac{\sqrt{m}}{4} + \frac{g}{2} \left(1 + 3 \log 2 - \log 3 + \left(\frac{bg}{m} - \frac{1}{2} \right) \log b - \frac{1}{2} (\log m + \log g) \right). \end{aligned} \quad (7.37)$$

Let the expression in parentheses be denoted by $f_2(b, q, m)$. Since f_2 is not necessarily non-negative, we first consider the subcase $f_2(b, q, m) < 0$.

(ii-1) The subcase $m \geq b$ and $f_2 < 0$.

When $f_2(b, q, m) < 0$, it follows from (7.37) that

$$\log S(b, q, m) < \frac{\sqrt{m}}{4}.$$

Substituting this into the right-hand side of (7.17) and denoting the resulting expression by $G(b, q, m)$, we obtain

$$G(b, q, m) = \log C_0 - m + \left(\frac{m}{2} + 1 \right) \log b + \frac{3}{2} \log q + \left(m - q(b-1) + \frac{1}{2} \right) \log m + \frac{\sqrt{m}}{4},$$

and

$$\begin{aligned} \frac{\partial^2 G}{\partial m^2} &= \frac{1}{m} + \frac{q(b-1) - \frac{1}{2} - \frac{1}{16}\sqrt{m}}{m^2} \geq \frac{q(b-1) - \frac{1}{2} - \frac{1}{16}\sqrt{\frac{bq}{2}}}{m^2} \\ &= \frac{\sqrt{q} \left(\sqrt{q}(b-1) - \frac{1}{16}\sqrt{\frac{b}{2}} \right) - \frac{1}{2}}{m^2} \geq \frac{\sqrt{2} \left(\sqrt{2}(b-1) - \frac{1}{16}\sqrt{\frac{b}{2}} \right) - \frac{1}{2}}{m^2} \\ &= \frac{\sqrt{2b} \left(\sqrt{2b} - \frac{1}{16\sqrt{2}} \right) - \sqrt{2} - \frac{1}{2}}{m^2} \geq \frac{\sqrt{8} \left(\sqrt{8} - \frac{1}{16\sqrt{2}} \right) - \sqrt{2} - \frac{1}{2}}{m^2} > 0. \end{aligned} \quad (7.38)$$

Thus G is convex in m . Since $b \leq m \leq bq/2$, the maximum is attained at one of the boundary points.

Setting $H_1(b, q) := G(b, q, b)$ and $H_2(b, q) := G(b, q, bq/2)$, we obtain

$$\begin{aligned} H_1(b, q) &= \log C_0 + \frac{\sqrt{b}}{4} - b + \left(-bq + \frac{3}{2}b + q + \frac{3}{2} \right) \log b + \frac{3}{2} \log q, \\ \frac{\partial H_1}{\partial b} &= \left(\frac{3}{2} - q \right) \log b + \left(\frac{1}{b} - 1 \right) q + \frac{1}{2} + \frac{3}{2b} + \frac{1}{8\sqrt{b}} \\ &\leq \left(\frac{3}{2} - 2 \right) \log 4 + \left(\frac{1}{4} - 1 \right) \cdot 2 + \frac{1}{2} + \frac{3}{2 \cdot 4} + \frac{1}{8\sqrt{4}} < 0, \\ \frac{\partial H_1}{\partial q} &= (1-b) \log b + \frac{3}{2q} \leq (1-4) \log 4 + \frac{3}{2 \cdot 2} < 0, \end{aligned}$$

and

$$\begin{aligned}
H_2(b, q) &= \log C_0 + \left(\frac{bq}{2} - q - \frac{1}{2} \right) \log 2 + \frac{\sqrt{2bq}}{8} - \frac{bq}{2} + \left(-\frac{bq}{4} + q + \frac{3}{2} \right) \log b \\
&\quad + \left(-\frac{bq}{2} + q + 2 \right) \log q, \\
\frac{\partial H_2}{\partial b} &= \frac{q}{4} \left(-\log b - 2 \log q - 3 + 2 \log 2 + \frac{4}{b} + \frac{1}{4} \sqrt{\frac{2}{bq}} \right) + \frac{3}{2b} \\
&\leq \frac{2}{4} \left(-\log 4 - 2 \log 2 - 3 + 2 \log 2 + \frac{4}{4} + \frac{1}{4} \sqrt{\frac{2}{4 \cdot 2}} \right) + \frac{3}{2 \cdot 4} < 0, \\
\frac{\partial H_2}{\partial q} &= \frac{b}{4} \left(-\log b - 4 + 2 \log 2 + \frac{1}{4} \sqrt{\frac{2}{bq}} + \frac{4 \log b}{b} \right) + \left(1 - \frac{b}{2} \right) \log q - \log 2 + 1 + \frac{2}{q} \\
&\leq \frac{4}{4} \left(-\log 4 - 4 + 2 \log 2 + \frac{1}{4} \sqrt{\frac{2}{4 \cdot 2}} + \frac{4 \log 4}{4} \right) + \left(1 - \frac{4}{2} \right) \log 2 - \log 2 + 1 + \frac{2}{2} \\
&< 0.
\end{aligned}$$

Thus, both H_1 and H_2 are decreasing with respect to both b and q .

Noting that q must be even whenever b is even, we compute the values as follows:

$$\begin{aligned}
H_1(4, 2) &> 0, \\
H_1(4, 4), H_1(5, 2) &< 0, \\
H_2(4, 2) &> 0, \\
H_2(4, 4), H_2(5, 2) &< 0.
\end{aligned}$$

For $(b, q) = (4, 2)$, we have $m = 2, 4$, and for those combinations $f_2(b, q, m)$ is always positive. Hence $(b, q) = (4, 2)$ does not fall into the present subcase, and we conclude that $G(b, q, m) < 0$ for all (b, q, m) with $m \geq b$ and $f_2(b, q, m) < 0$.

(ii-2) The subcase $m \geq b$ and $f_2 \geq 0$.

Since $f_2(b, q, m) \geq 0$, applying the bound $2 \leq g \leq m/2$ to (7.37), we obtain

$$\begin{aligned}
\log S(b, q, m) &\leq \frac{\sqrt{m}}{4} + \frac{m}{4} \left(1 + 3 \log 2 - \log 3 + \left(\frac{bq}{m} - \frac{1}{2} \right) \log b - \frac{1}{2} (\log m + \log 2) \right) \\
&= \frac{\sqrt{m}}{4} + \frac{m}{4} \left(1 + \frac{5}{2} \log 2 - \log 3 + \left(\frac{bq}{m} - \frac{1}{2} \right) \log b - \frac{1}{2} \log m \right).
\end{aligned}$$

Substituting this into the right-hand side of (7.17) and denoting the resulting expression by $G(b, q, m)$, we obtain

$$\begin{aligned}
G(b, q, m) &= \log C_0 - m + \left(\frac{m}{2} + 1 \right) \log b + \frac{3}{2} \log q + \left(m - q(b-1) + \frac{1}{2} \right) \log m \\
&\quad + \frac{\sqrt{m}}{4} + \frac{m}{4} \left(1 + \frac{5}{2} \log 2 - \log 3 + \left(\frac{bq}{m} - \frac{1}{2} \right) \log b - \frac{1}{2} \log m \right), \quad (7.39)
\end{aligned}$$

and by the same type of estimate as in (7.38),

$$\frac{\partial^2 G}{\partial m^2} = \frac{7}{8m} + \frac{q(b-1) - \frac{1}{2} - \frac{1}{16} \sqrt{m}}{m^2} \geq \frac{q(b-1) - \frac{1}{2} - \frac{1}{16} \sqrt{\frac{bq}{2}}}{m^2} > 0.$$

Thus G is convex in m . Since $b \leq m \leq bq/2$, the maximum is attained at one of the boundary points.

Setting $H_1(b, q) := G(b, q, b)$ and $H_2(b, q) := G(b, q, bq/2)$, we obtain

$$G(b, q, m) \leq \max\{H_1(b, q), H_2(b, q)\}.$$

Substituting $m = b$ into (7.39), we obtain

$$H_1(b, q) = \log C_0 + \frac{\sqrt{b}}{4} - \frac{3}{4}b + \frac{5}{8}b \log 2 - \frac{b}{4} \log 3 + \left(-\frac{3}{4}bq + \frac{5}{4}b + q + \frac{3}{2}\right) \log b + \frac{3}{2} \log q,$$

$$\begin{aligned} \frac{\partial H_1}{\partial b} &= \frac{-3q+5}{4} \log b + \left(-\frac{3}{4} + \frac{1}{b}\right)q + \frac{5}{8} \log 2 - \frac{1}{4} \log 3 + \frac{1}{2} + \frac{3}{2b} + \frac{1}{8\sqrt{b}} \\ &\leq \frac{-3 \cdot 2 + 5}{4} \log 4 + \left(-\frac{3}{4} + \frac{1}{4}\right) \cdot 2 + \frac{5}{8} \log 2 - \frac{1}{4} \log 3 + \frac{1}{2} + \frac{3}{2 \cdot 4} + \frac{1}{8\sqrt{4}} < 0, \end{aligned}$$

$$\frac{\partial H_1}{\partial q} = \left(1 - \frac{3}{4}b\right) \log b + \frac{3}{2q} \leq \left(1 - \frac{3}{4} \cdot 4\right) \log 4 + \frac{3}{2 \cdot 2} < 0.$$

Substituting $m = bq/2$ into (7.39), we obtain

$$\begin{aligned} H_2(b, q) &= \log C_0 + \left(\frac{7}{8}bq - q - \frac{1}{2}\right) \log 2 - \frac{bq}{8} \log 3 - \frac{3}{8}bq + \frac{\sqrt{2bq}}{8} \\ &\quad + \left(-\frac{bq}{8} + q + \frac{3}{2}\right) \log b + \left(-\frac{9}{16}bq + q + 2\right) \log q, \\ \frac{\partial H_2}{\partial b} &= \frac{q}{16} \left(-2 \log b - 9 \log q - 8 + 14 \log 2 - 2 \log 3 + \frac{16}{b} + \sqrt{\frac{2}{bq}}\right) + \frac{3}{2b} \\ &\leq \frac{2}{16} \left(-2 \log 4 - 9 \log 2 - 8 + 14 \log 2 - 2 \log 3 + \frac{16}{4} + \sqrt{\frac{2}{4 \cdot 2}}\right) + \frac{3}{2 \cdot 4} < 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial H_2}{\partial q} &= \frac{b}{16} \left(-2 \log b - 15 + 14 \log 2 - 2 \log 3 + \sqrt{\frac{2}{bq}} + \frac{16 \log b}{b}\right) \\ &\quad + 1 - \log 2 + \frac{2}{q} + \left(-\frac{9}{16}b + 1\right) \log q \\ &\leq \frac{4}{16} \left(-2 \log 4 - 15 + 14 \log 2 - 2 \log 3 + \sqrt{\frac{2}{4 \cdot 2}} + \frac{16 \log 4}{4}\right) \\ &\quad + 1 - \log 2 + \frac{2}{2} + \left(-\frac{9}{16} \cdot 4 + 1\right) \log 2 < 0. \end{aligned}$$

Thus, both H_1 and H_2 are decreasing with respect to both b and q .

Noting that q must be even whenever b is even, we compute the values as follows:

$$\begin{aligned} H_1(k, 2) \quad (4 \leq k \leq 9) &> 0, \\ H_1(4, 4), H_1(5, 3), H_1(10, 2) &< 0, \\ H_2(k, 2) \quad (4 \leq k \leq 9), H_2(4, 4), H_2(5, 3) &> 0, \\ H_2(4, 6), H_2(5, 4), H_2(7, 3), H_2(10, 2) &< 0. \end{aligned}$$

Therefore, we have $G(b, q, m) < 0$ except for

$$(b, q) = (k, 2) \quad (4 \leq k \leq 9), (4, 4), (5, 3).$$

Combining the results of (i), (ii-1), and (ii-2), when $|D_r| \geq 2$, $\beta \geq g/2$, and $m \nmid q$, the inequality (7.14) holds for all pairs except the following:

$$(b, q) = (k, 2) \quad (4 \leq k \leq 9), (4, 4), (5, 3). \quad (7.40)$$

By combining all the five cases (7.23), (7.24), (7.28), (7.32), and (7.40), we have shown that (7.14) holds for all pairs (b, q) except the following:

$$\begin{aligned} &(3, k) \quad (2 \leq k \leq 20), \\ &(4, 2), (4, 4), (4, 6), (4, 8), \\ &(5, 2), (5, 3), (5, 4), (5, 5), \\ &(6, 2), (7, 2), (8, 2), (9, 2). \end{aligned}$$

For each of these exceptional pairs, direct computation shows that there exists a permutation y such that, with $x = (1 \ 2 \ \dots \ n)$ fixed, the automorphism group is trivial; see Table 3 and Table 4.

When $(b, q) = (3, 2)$, the monodromy group $G = \langle x, y \rangle$ has order 120 and $C_{S_n}(G) \cong \{1\}$. For all remaining (b, q) pairs, the group $\langle x, y \rangle$ is equal to S_n or A_n . Since $C_{S_n}(S_n) \cong C_{S_n}(A_n) \cong \{1\}$, the automorphism group is trivial for each such pair (x, y) .

Thus, we conclude that every passport $[n, b^q, n]$ of genus ≥ 2 with $b \geq 3$ and $q \geq 2$ admits at least one dessin with a trivial automorphism group. $\square$

To find elements y such that $\langle x, y \rangle = S_n$ or A_n for $x = (1 \ 2 \ \dots \ n)$, we applied Theorem 7.4, as in the case of $b = 2$.

For a permutation y of cycle type (b^q) , assume that $G = \langle x, y \rangle$ has no blocks with respect to any residue class of a divisor m of $n = bq$ with $2 \leq m < n$, and that G contains an element w of cycle type (p_w) for a prime $p_w \leq n - 3$. Then $G = S_n$ when n is even, and $G = A_n$ when n is odd.

A list of such examples of y and w obtained by computation is shown in Table 3 and Table 4. As in Table 2, we completed this table by generating random permutations y of cycle type (b^q) and searching for an element w satisfying the required conditions among the elements of $\langle x, y \rangle$.

When $(b, q) = (3, 2)$, the monodromy group never becomes S_6 . Instead, it can be a group of order 120, in which case the automorphism group is trivial. A dessin realizing this case is shown in Figure 10. This is the same dessin as the one shown in the lower right of Figure 5 in Section 1.3.

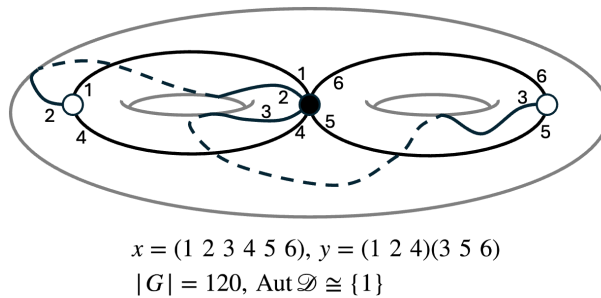


FIGURE 10. A dessin with trivial automorphism group having passport $[6, 3^2, 6]$

7.5. Passports $[n, b^q, n]$

Combining the results of this section, we obtain the following theorem.

Theorem 7.6. *If a uniform passport $[n, b^q, n]$ ($q \geq 1$) has genus at least 2, then it admits a dessin d'enfant with a trivial automorphism group.*

The same statement holds for the passports $[b^q, n, n]$ and $[n, n, b^q]$.

b	q	y			
		G	$w(x, y)$	p_w	w
3	2	(1 2 4)(3 5 6)			
		120			
3	3	(1 2 4)(3 6 8)(5 9 7)			
		A_9	$x^3 y x y^2 x^2 y^2$	5	(1 4 9 5 2)
3	4	(1 10 11)(2 4 7)(3 5 9)(6 12 8)			
		S_{12}	$y x y x^2 y x y x^2$	7	(1 9 10 11 12 4 3)
3	5	(1 3 6)(2 10 8)(4 7 9)(5 11 15)(12 13 14)			
		A_{15}	$x^2 y x y^2 x^2 y^2 x$	11	(1 8 4 6 9 10 13 15 2 12 14)
3	6	(1 13 10)(2 7 4)(3 16 9)(5 15 11)(6 14 17)(8 12 18)			
		S_{18}	$x^4 y x y x^2 y^2 x^3$	13	(1 6 7 4 2 9 11 14 5 16 17 15 12)
3	7	(1 13 16)(2 6 7)(3 4 5)(8 20 18)(9 19 21)(10 12 17)(11 14 15)			
		A_{21}	$x y^2 x^2 y x^2 y$	17	(1 2 17 10 13 18 12 8 15 21 14 11 6 20 9 16 3)
3	8	(1 17 22)(2 8 24)(3 15 23)(4 6 20)(5 12 18)(7 21 9)(10 14 16)(11 19 13)			
		S_{24}	$x y x^2 y x^2 y^2 x y x y x$	19	(2 18 23 19 20 17 8 7 6 15 9 11 5 22 10 21 13 3 24)
3	9	(1 26 24)(2 9 22)(3 17 15)(4 12 27)(5 20 8)(6 21 19)(7 13 10)(11 18 14)(16 23 25)			
		A_{27}	$x^2 y x^2 y x^2 y$	17	(2 10 3 7 22 13 11 9 19 15 4 12 20 24 8 5 14)
3	10	(1 13 30)(2 8 29)(3 5 7)(4 28 12)(6 21 14)(9 19 24)(10 22 23)(11 16 26)(15 27 20)(17 18 25)			
		S_{30}	$y x^2 y^2 x y x^2 y^2 x$	23	(1 10 5 18 20 27 4 28 9 24 30 11 13 12 26 16 29 21 7 22 17 15 8)
3	11	(1 24 12)(2 27 32)(3 20 31)(4 29 26)(5 19 14)(6 33 10)(7 15 11)(8 21 22)(9 18 28)(13 16 23)(17 25 30)			
		A_{33}	$x^2 y^2 x^2 y x^3 y^2 x$	29	(1 6 13 2 31 7 29 17 33 25 28 32 5 11 14 24 8 16 26 10 27 3 20 30 9 23 22 15 19)
3	12	(1 15 12)(2 29 25)(3 17 18)(4 26 35)(5 16 13)(6 34 8)(7 30 27)(9 31 14)(10 24 33)(11 20 23)(19 21 36)(22 28 32)			
		S_{36}	$x y x y x^6 y^2$	31	(1 27 24 10 4 19 5 29 31 6 25 17 23 20 22 28 11 36 7 21 18 2 13 26 3 9 34 30 35 12 16)
3	13	(1 26 18)(2 12 5)(3 11 15)(4 30 31)(6 21 13)(7 19 14)(8 32 34)(9 17 25)(10 37 29)(16 24 33)(20 27 35)(22 28 39)(23 36 38)			
		A_{39}	$x y^2 x^2 y^2 x^2 y^2 x$	31	(1 34 29 12 19 3 2 4 7 18 21 15 38 14 39 11 25 10 20 24 26 6 28 5 22 36 32 16 13 35 8)
3	14	(1 41 22)(2 40 4)(3 16 33)(5 31 10)(6 35 21)(7 36 13)(8 30 12)(9 20 15)(11 24 17)(14 23 37)(18 25 42)(19 39 26)(27 28 29)(32 34 38)			
		S_{42}	$x y^2 x y x^2 y^2 x^2 y$	37	(2 29 37 39 28 24 9 26 21 12 41 13 31 7 38 20 10 32 36 5 40 35 33 16 6 23 14 4 27 22 42 19 25 8 11 15 3)
3	15	(1 40 5)(2 29 6)(3 19 9)(4 21 27)(7 26 45)(8 28 34)(10 16 33)(11 23 41)(12 20 25)(13 37 24)(14 15 18)(17 38 42)(22 35 30)(31 32 36)(39 43 44)			
		A_{45}	$y^2 x^2 y^2 x^2$	41	(1 41 23 30 16 42 26 33 37 5 6 32 35 14 25 11 7 4 36 3 38 9 39 20 31 15 43 8 22 44 45 34 28 13 10 21 24 19 29 17 40)
3	16	(1 4 14)(2 33 20)(3 8 22)(5 24 46)(6 36 32)(7 47 16)(9 27 41)(10 34 30)(11 12 23)(13 45 37)(15 48 35)(17 43 26)(18 39 28)(19 44 42)(21 38 29)(25 40 31)			
		S_{48}	$y x y x y^2 x y x y x y^2 x$	43	(1 21 41 7 4 8 23 40 33 9 3 34 47 5 37 20 10 19 26 32 6 31 28 18 29 48 43 2 13 44 45 17 30 24 22 38 14 12 16 25 15 27 42)
3	17	(1 40 23)(2 32 9)(3 24 22)(4 46 34)(5 31 11)(6 12 18)(7 51 39)(8 33 16)(10 21 29)(13 30 41)(14 25 19)(15 50 17)(20 43 47)(26 27 36)(28 42 48)(35 38 49)(37 44 45)			
		A_{51}	$x y^2 x^3 y x y x y x$	47	(1 13 45 30 51 49 23 50 15 27 6 34 28 18 29 36 24 43 31 19 46 38 40 14 37 33 7 47 17 21 41 22 26 2 5 20 8 25 16 12 11 10 9 39 35 44 48)
3	18	(1 4 28)(2 20 23)(3 25 11)(5 42 54)(6 7 38)(8 45 47)(9 34 21)(10 52 40)(12 32 29)(13 37 35)(14 26 27)(15 44 17)(16 51 22)(18 19 43)(24 53 36)(30 41 50)(31 33 48)(39 46 49)			
		S_{54}	$y x y^2 x y^2 x y^2 x y$	47	(1 48 5 32 47 3 17 13 21 39 36 53 42 29 18 41 34 27 51 6 52 8 11 35 4 37 23 22 16 44 2 45 50 49 28 26 12 24 25 15 40 19 9 31 33 10 20)
3	19	(1 33 50)(2 20 34)(3 36 19)(4 56 25)(5 32 38)(6 22 30)(7 15 57)(8 53 41)(9 12 47)(10 39 14)(11 55 52)(13 26 45)(16 43 28)(17 44 24)(18 49 37)(21 27 29)(23 42 40)(31 35 46)(48 54 51)			
		A_{57}	$x^3 y x^5 y x y^2$	53	(1 44 54 43 40 53 47 38 55 14 19 45 5 6 26 3 17 15 36 2 51 10 35 31 13 22 37 25 50 18 42 33 7 8 16 48 46 20 11 34 41 28 24 52 9 23 29 30 12 27 49 56 21)
3	20	(1 6 35)(2 8 17)(3 46 54)(4 14 29)(5 50 48)(7 51 52)(9 28 26)(10 55 56)(11 34 59)(12 24 36)(13 57 37)(15 42 19)(16 23 60)(18 21 31)(20 22 45)(25 53 33)(27 32 39)(30 49 40)(38 47 41)(43 44 58)			
		S_{60}	$y x y^2 x y^2 x y^2 x y$	53	(1 4 14 52 34 47 12 13 2 48 39 29 45 11 9 54 5 56 57 6 24 60 30 21 37 18 44 33 22 38 46 36 23 59 20 32 8 55 40 15 51 42 49 17 16 19 53 43 28 35 26 50 31)
4	2	(1 6 2 7)(3 4 8 5)			
		S_8	$y^2 x y^2 x$	3	(3 7 5)
4	4	(1 4 8 15)(2 3 10 16)(5 14 7 12)(6 13 11 9)			
		S_{16}	$y^3 x y x$	13	(2 13 4 8 14 16 12 7 10 3 11 9 15)
4	6	(1 20 11 6)(2 24 15 13)(3 23 18 22)(4 17 7 8)(5 10 16 14)(9 19 12 21)			
		S_{24}	$x^2 y^2 x^4 y^2 x^2$	19	(2 3 8 5 19 24 23 4 15 11 9 18 16 6 21 17 13 22 10)
4	8	(1 29 8 28)(2 15 26 14)(3 32 9 13)(4 22 31 18)(5 27 6 24)(7 17 19 12)(10 25 30 20)(11 23 21 16)			
		S_{32}	$x^2 y^2 x^3 y^2 x$	29	(1 30 21 13 24 18 32 23 12 11 27 15 4 5 3 28 6 20 17 22 9 10 29 2 19 31 25 8 7)

TABLE 3. Elements y such that $\text{Aut } \mathscr{D} \cong \{1\}$ for $b = 3, 4$ ($x = (1 \ 2 \ \dots \ n)$)

b	q	y			
		G	$w(x, y)$	p_w	w
5	2	(1 2 10 8 5)(3 7 4 9 6)			
		S_{10}	y^2xy^2x	7	(1 3 9 7 8 6 5)
5	3	(1 11 12 5 8)(2 9 10 3 4)(6 14 7 13 15)			
		A_{15}	$y^2x^2y^3x$	11	(2 5 14 3 8 7 13 10 4 6 11)
5	4	(1 8 2 16 13)(3 19 17 10 20)(4 11 15 7 12)(5 6 9 14 18)			
		S_{20}	xyx^2y^3x	17	(1 20 6 2 5 4 14 10 19 18 17 16 7 8 13 3 15)
5	5	(1 8 19 24 21)(2 13 7 12 17)(3 6 18 4 9)(5 15 22 25 11)(10 16 23 20 14)			
		A_{25}	$xy^2xy^2xy^2xy^2$	17	(1 21 10 25 15 16 18 8 23 11 7 2 5 20 9 24 6)
6	2	(1 9 5 7 10 6)(2 3 4 12 8 11)			
		S_{12}	x^3yx^4yx	3	(4 5 10)
7	2	(1 4 14 11 12 6 9)(2 10 3 8 5 13 7)			
		S_{14}	y^2xy^2x	11	(1 11 8 7 12 5 3 9 4 13 2)
8	2	(1 14 4 9 11 15 8 13)(2 7 10 5 16 12 3 6)			
		S_{16}	y^2y^4	13	(2 7 4 5 13 15 12 11 6 10 16 9 8)
9	2	(1 10 3 15 9 5 2 12 17)(4 11 13 18 16 14 7 8 6)			
		S_{18}	$x^3yx^2y^2x^2yx$	13	(1 6 4 12 13 14 15 8 9 10 11 5 3)

TABLE 4. Elements y such that $\text{Aut } \mathcal{D} \cong \{1\}$ for $b \geq 5$ ($x = (1 \ 2 \ \dots \ n)$)

Proof. Since the automorphism group is symmetric with respect to black vertices, white vertices, and faces, it suffices to prove the statement for the passport $[n, b^q, n]$ ($n = bq$).

By (2.4), the genus is

$$\frac{n - (1 + q + 1)}{2} + 1 = \frac{n - q}{2} = \frac{q(b - 1)}{2} \geq 2.$$

Therefore, we have $b \geq 2$.

The case $b = n$ ($q = 1$) was proved in Proposition 7.1.

The case $b = 2$ was proved in Proposition 7.3.

The case $3 \leq b < n$ ($q \geq 2$) was proved in Proposition 7.5. $\square$

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