

Regularity and Automorphism Groups of Dessins d'Enfants with Uniform Passports

Tatsuya Ohnishi (大西 達也) The University of Osaka March 10, 2026

Belyĭ's Theorem (G. V. Belyĭ, 1979)

For a compact Riemann surface X , (1) $\Leftrightarrow$ (2).

(1) X can be defined over $\bar{\mathbb{Q}}$ (the algebraic closure of $\mathbb{Q}$)

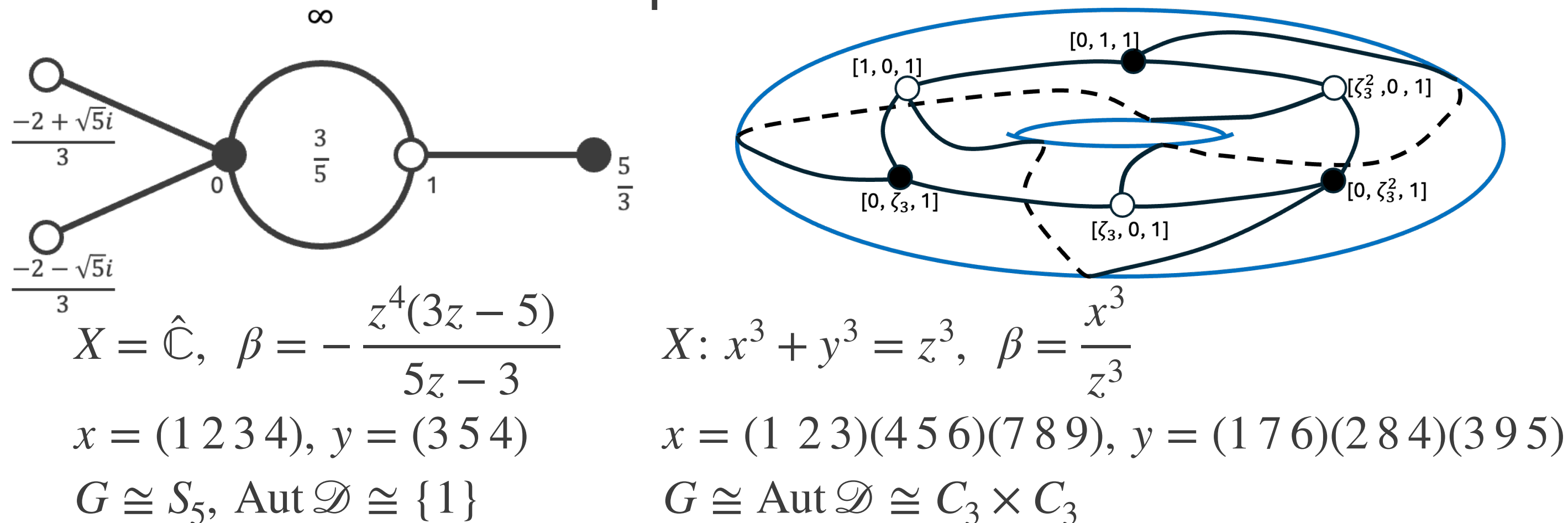
(2) There exists a non-constant meromorphic function

$\beta: X \rightarrow \hat{\mathbb{C}}$ ramified over at most three points

β : Belyĭ function, (X, β) : Belyĭ pair can be taken to be 0, 1, and ∞

Dessin d'Enfant

- A connected bipartite graph associated with (X, β)
- Drawn on a closed surface of the same genus as X
- $\bullet$: $\beta^{-1}(0)$, $\circ$: $\beta^{-1}(1)$, faces: $\beta^{-1}(\infty)$, edges: $\beta^{-1}([0,1])$
- Each face is homeomorphic to a disk



Regularity & Automorphism Groups

- Passport** of a dessin:

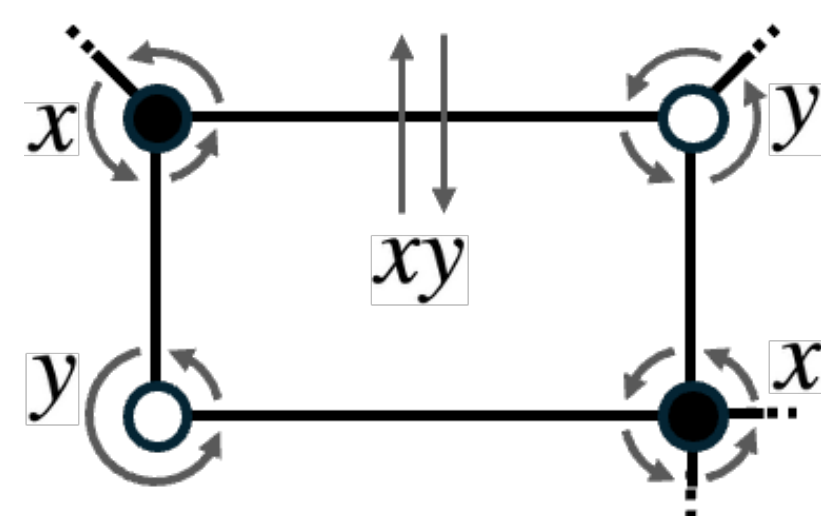
A triple $[\lambda_0, \lambda_1, \lambda_\infty]$ of partitions $\lambda_0, \lambda_1, \lambda_\infty \vdash n$ (= number of edges) corresponding resp. to valencies of $\bullet$, $\circ$, faces

- A passport / dessin is **uniform**

$\Leftrightarrow$ Valencies of $\bullet$, $\circ$, and faces are each constant

- Monodromy group** G of a dessin:

For the permutations of the edges shown on the right, $G = \langle x, y \rangle$



- Aut $\mathcal{D}$: Automorphism group** of a dessin $\mathcal{D}$:

The group of permutations of the edges that preserves the cyclic order of edges around each $\bullet$, $\circ$

- A dessin is **regular** (i.e., has the highest symmetry)

$\Leftrightarrow G$ acts freely and transitively on the edges

$\Leftrightarrow \text{Aut } \mathcal{D}$ acts freely and transitively on the edges

$\Leftrightarrow$ The Belyĭ function $\beta: X \rightarrow \hat{\mathbb{C}}$ is a Galois cover

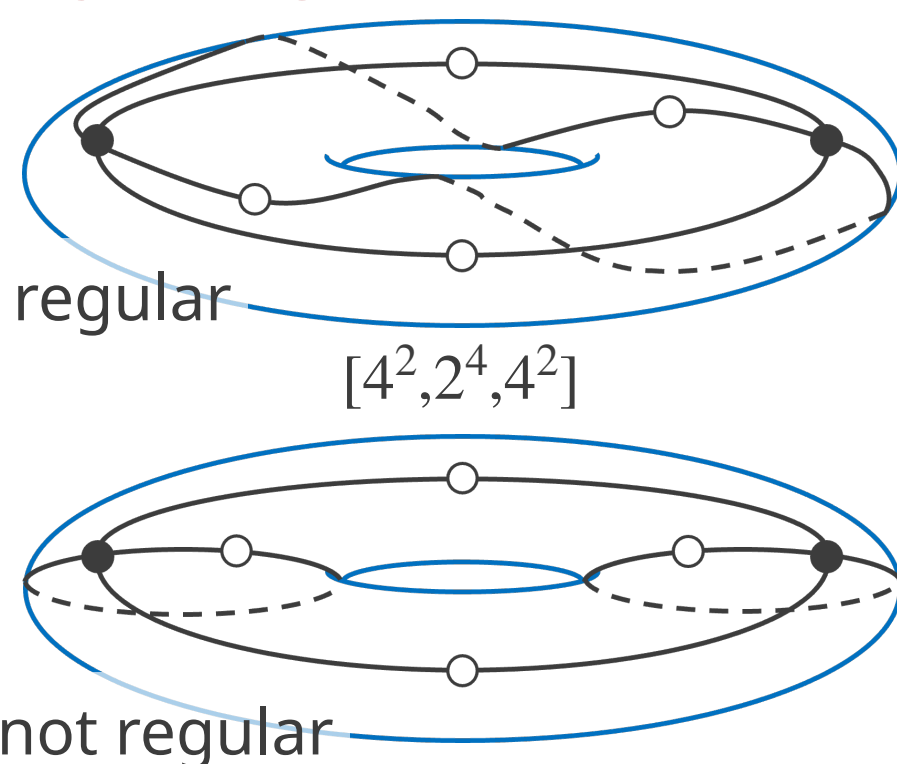
$\Leftrightarrow \text{Aut } \mathcal{D} \cong G$

Main Problems

Fact: A dessin is regular $\not\Leftrightarrow$ A dessin is uniform

Questions: For a uniform passport,

- When do regular dessins exist, and how many?
- Distribution of automorphism groups
- Genus dependence



Conjectural Picture

Genus	$\text{Aut } \mathcal{D} \cong \{1\}$	$1 < \text{Aut } \mathcal{D} < n$	$ \text{Aut } \mathcal{D} = n$ (regular)
0	– (1)	– (1)	$\checkmark^{(1)}$
1 (with an (n) -cycle)	– (3)	– (2)	$\checkmark^{(5)}$
1 (without an (n) -cycle)	– (3)	$\checkmark^{(2)}$	$\diamond$
≥ 2 (with ≥ 2 (n) -cycles)	$\checkmark^{(4)}$	$\diamond$	$\checkmark^{(5)}$
≥ 2 (with ≤ 1 (n) -cycle)	$\checkmark$	$\checkmark$	$\diamond$

$\checkmark$: always occurs, $-$: never occurs, $\diamond$: depends on the passport

Main Results

We prove (1)–(5). In particular:

(4) Every uniform passport $[n, b^q, n]$ of genus at least 2 admits a dessin with trivial automorphism group.

(5) A uniform passport $[a^p, b^q, n]$ admits a regular dessin if and only if $\gcd(p, q) = 1$.

Sketch of the Proof of (4)

◆ The case $q = 1$ (passport $[n, n, n]$, n : odd, $n \geq 5$)

Let $x = (1\ 2\ \dots\ n)$, $y = (2\ 4\ 6\ \dots\ n-1\ n\ n-2\ \dots\ 3\ 1)$

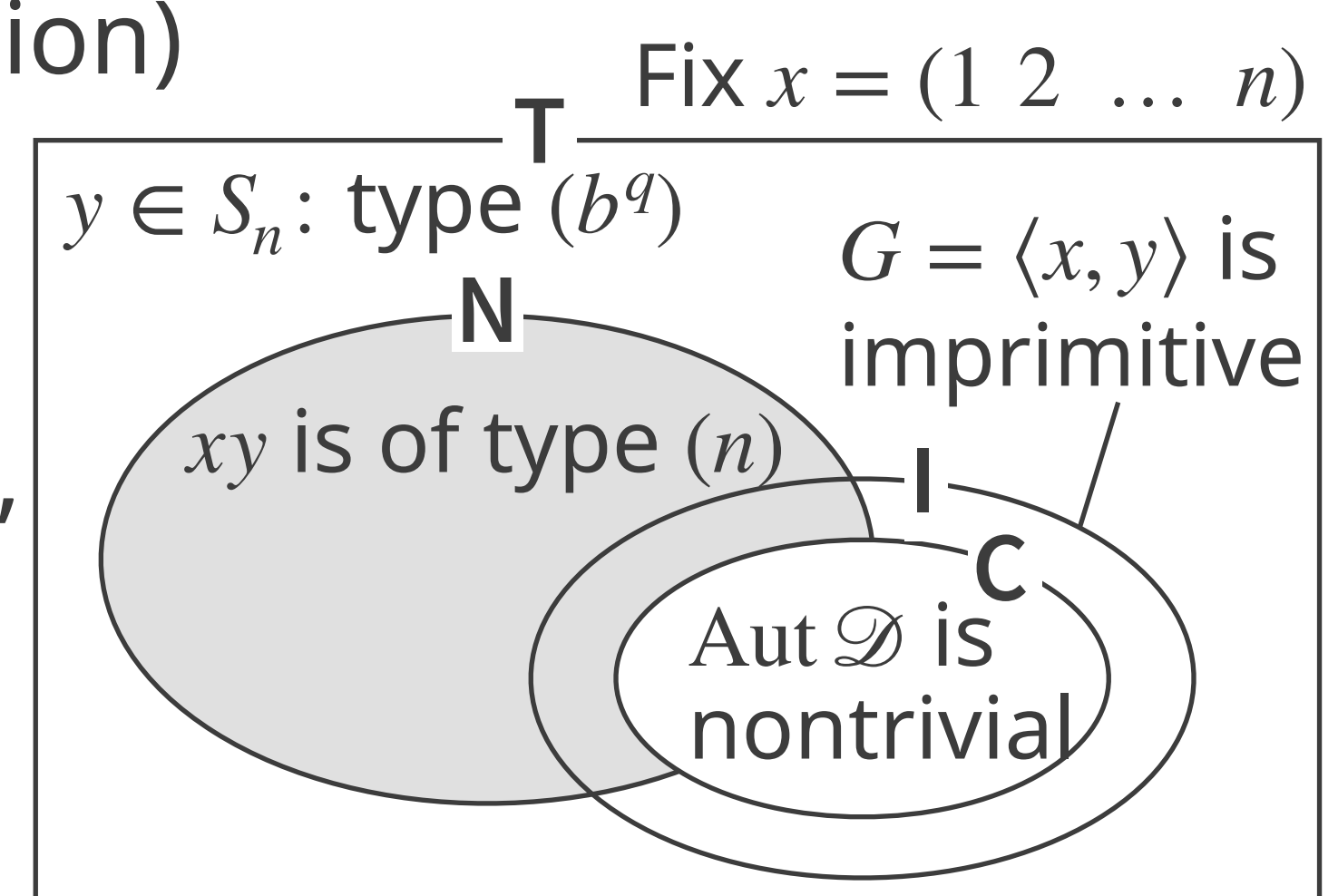
$\Rightarrow xy = (1\ 3\ 2\ 5\ 4\ \dots\ n-2\ n-3\ n\ n-1)$, $x^2(xyxy)^{-1} = (1\ 2\ 3)$, $\langle x, (1\ 2\ 3) \rangle = A_n$ (alternating group), $\text{Aut } \mathcal{D} \cong C_{S_n}(A_n) \cong \{1\}$

◆ The case $q \geq 2$

Key idea:

If G is primitive on the edges, then $\text{Aut } \mathcal{D}$ is trivial.

Existence of a dessin with trivial automorphism group
 $\Leftrightarrow N \cap C^c \neq \emptyset$ (the gray region)
 $\Leftrightarrow N/T > I/T$



- Lower bound of N/T

By [1] and further evaluation,

we obtain: $\frac{N}{T} \geq \frac{2}{n+2}$

- Upper bound of I/T

The rate of y for which G is imprimitive with respect to the partition of $\{1, \dots, n\}$ into residue classes modulo m ($m \mid n$, $2 \leq m < n$):

$$\frac{I_m}{T} = \frac{q!}{n!} m! \left(\left(\frac{n}{m} \right)! \right)^m \sum_{\{(d_i, t_i)\}} \prod_i \frac{1}{d_i^{t_i} t_i!} \left(\frac{d_i^{d_i q}}{\left(\frac{d_i q}{m} \right)!} \right)^{t_i}$$

Summation taken over all partitions $\{(d_i, t_i)\}$ of m satisfying $\sum_i d_i t_i = m$, $d_i \mid b$, and $m \mid d_i q$.

Since the number of divisors of n does not exceed $2\sqrt{n}$, it suffices to show $\forall m, H := \log \left(2\sqrt{n} \cdot \frac{n+2}{2} \cdot \frac{I_m}{T} \right) < 0$.

Balancing bound sharpness and case complexity,

- $b = 2$: split into $m = 2$ and $3 \leq m \leq q$;
- $b \geq 3$: split into five cases depending on b, q, m .

For each case we obtain a closed-form analytic upper bound $\hat{H}(b, q, m)$ of H , and for each of them,

- Show that $\hat{H}$ is convex as a function of m
- Show that the endpoints are decreasing in b and q
- This implies that the desired inequality holds for all but finitely many pairs (b, q)

$\rightarrow$ It remains to check 40 exceptional pairs (b, q)

We constructed explicit permutations y with trivial automorphism group by random computational search.
 $\rightarrow$ All cases are resolved. $\square$

References

[1] A. Goupil and G. Schaeffer, *Factoring N -cycles and counting maps of given genus*, European J. Combin. **19** (1998), no. 7, 819-834.