

Regularity and Automorphism Groups of Dessins d'Enfants with Uniform Passports

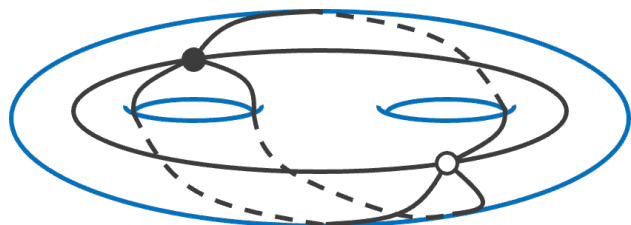
(一様なパスポートを持つdessin d'enfantの
正則性と自己同型群)

arXiv:2602.11867, arXiv:2606.27118

Tatsuya Ohnishi (大西 達也)

The University of Osaka

July 7, 2026



Belyĭ's Theorem & Dessin d'Enfant

Belyĭ's Theorem

Theorem. [G. V. Belyĭ, 1979]

For a compact Riemann surface X , the following are equivalent.

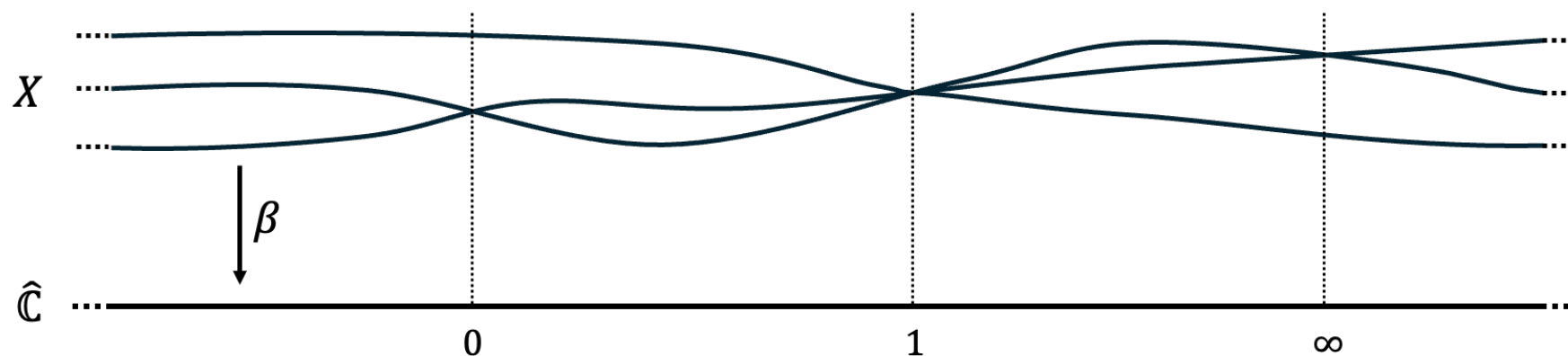
(1) X can be defined over $\overline{\mathbb{Q}}$

$$\begin{aligned} &\checkmark y^2 = x(x-1)(x+1) \cong y^2 = x(x-e)(x+e) & \times y^2 = x(x-1)(x-e) \\ &\checkmark y^2 = x(x-\sqrt{2})(x-\sqrt[3]{3}) \end{aligned}$$

(2) There exists a non-constant meromorphic function

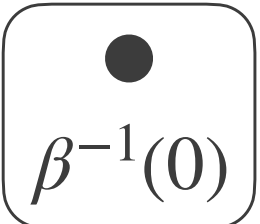
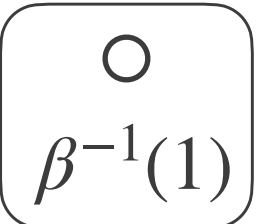
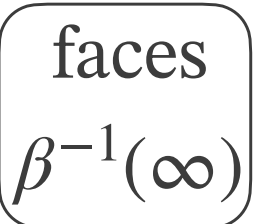
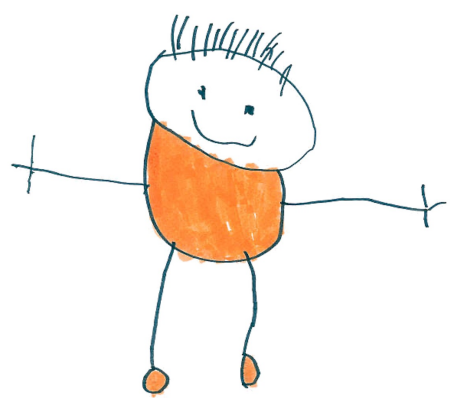
$\beta: X \rightarrow \hat{\mathbb{C}}$ ramified over at most three points $\leftarrow$ can be taken

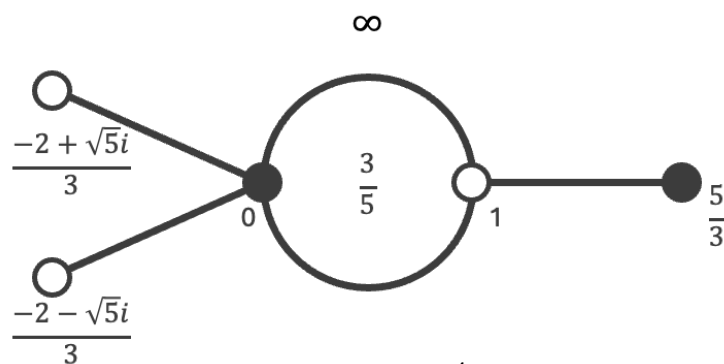
• β : **Belyĭ function** of X , (X, β) : **Belyĭ pair** to be 0, 1, ∞



$\rightarrow$ **dessin d'enfant** (A. Grothendieck, 1984)

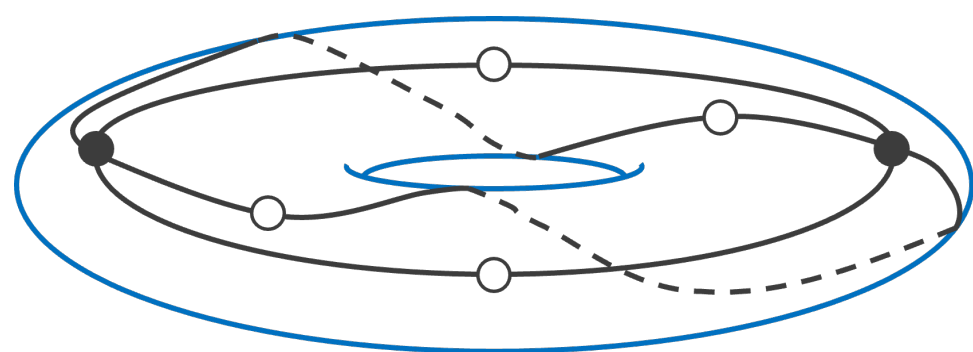
Dessin d'Enfant (child's drawing)

- A connected bipartite graph associated with (X, β) 
- Drawn on a closed surface of the same genus as X
-  $\beta^{-1}(0)$  $\beta^{-1}(1)$  $\beta^{-1}(\infty) \supset$ Critical points of X 
- Edges: $\beta^{-1}([0, 1])$
- Each face is homeomorphic to a disk



$$X = \hat{\mathbb{C}}, \quad \beta = -\frac{z^4(3z-5)}{5z-3}$$

Genus 0



$$X: y^2 = x(x-1)(x+1), \quad \beta = -\frac{(x-1)^2(x+1)^2}{4x^2}$$

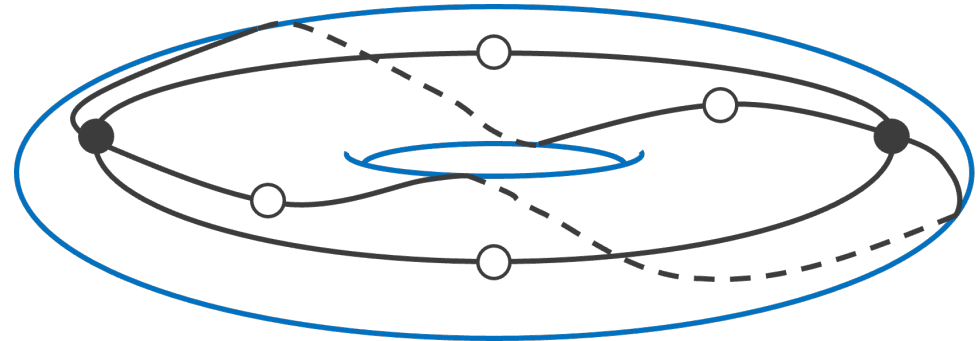
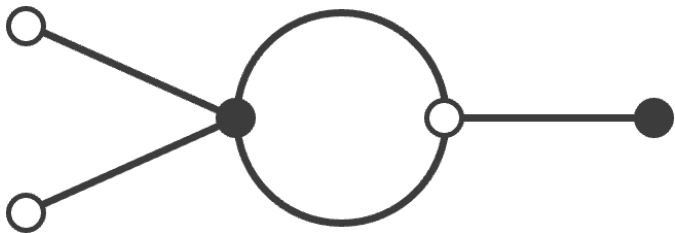
Genus 1

Definitions & Terminology

Passport and Uniformity

- **Passport (パスポート)** of a dessin:
The ordered list of valencies of $\bullet$, $\circ$, and faces
- **Uniform (一様な)** passport / dessin:
The valencies of $\bullet$, $\circ$, and faces are each constant

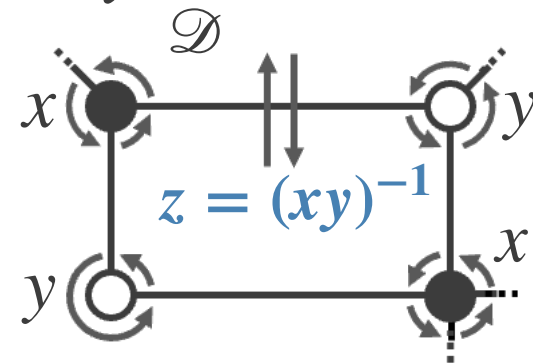
Passport: $[4\ 1, 3\ 1\ 1, 4\ 1]$ (not uniform) Passport: $[4^2, 2^4, 4^2]$ (uniform)



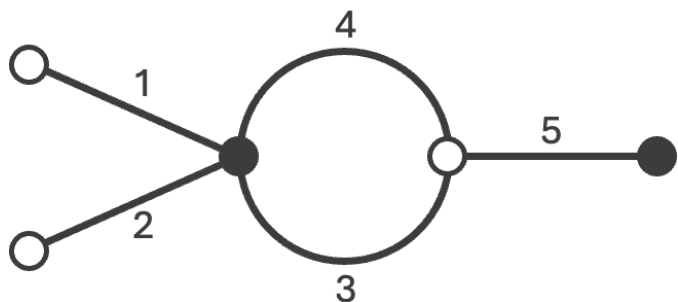
Monodromy Group and Automorphism Group

Counterclockwise edge permutation around $\bullet: x$, $\circ: y$

- $G = \langle x, y \rangle$: **monodromy group** (モノドロミー群)
 - $\text{Aut } \mathcal{D}$: **automorphism group** (自己同型群)
preserving the cyclic order around vertices
- $\text{Aut } \mathcal{D} = C_{S_n}(G) = \{c \in S_n \mid cx = xc, cy = yc\}$ (centralizer)

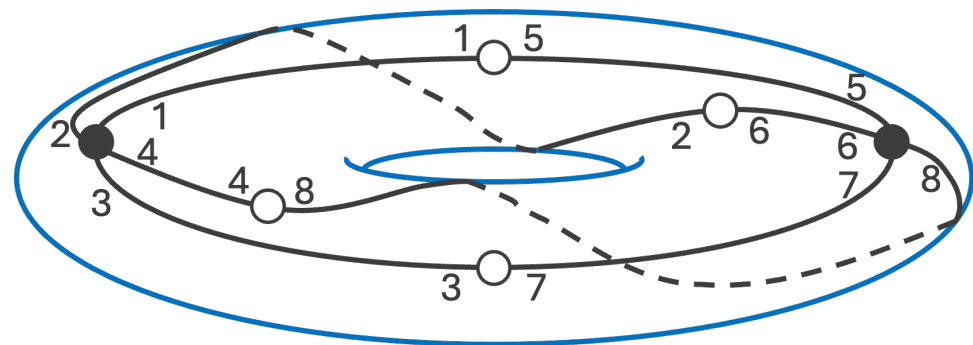


Reflect the complexity and symmetry of the dessin



$$x = (1 \ 2 \ 3 \ 4), \ y = (3 \ 5 \ 4)$$

$$G \cong S_5, \ \text{Aut } \mathcal{D} \cong \{1\}$$



$$x = (1 \ 2 \ 3 \ 4)(5 \ 6 \ 7 \ 8), \ y = (1 \ 5)(2 \ 6)(3 \ 7)(4 \ 8)$$

$$G \cong \text{Aut } \mathcal{D} \cong C_4 \times C_2$$

Regularity of Dessins

A dessin $\mathcal{D}$ is **regular** (正則) (maximally symmetric)

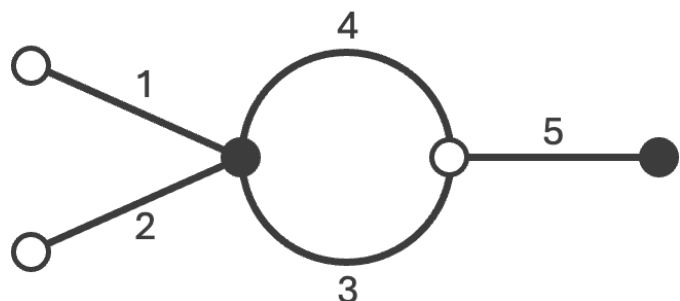
$\stackrel{\text{def}}{\iff}$ the monodromy group G acts freely on the edge set E

$\iff$ the automorphism group $\text{Aut } \mathcal{D}$ acts transitively on E

$\iff$ the Belyi function $\beta: X \rightarrow \hat{\mathbb{C}}$ is a Galois covering

$\iff \text{Aut } \mathcal{D} \cong G \iff |\text{Aut } \mathcal{D}| = |G| = n$ (number of edges)

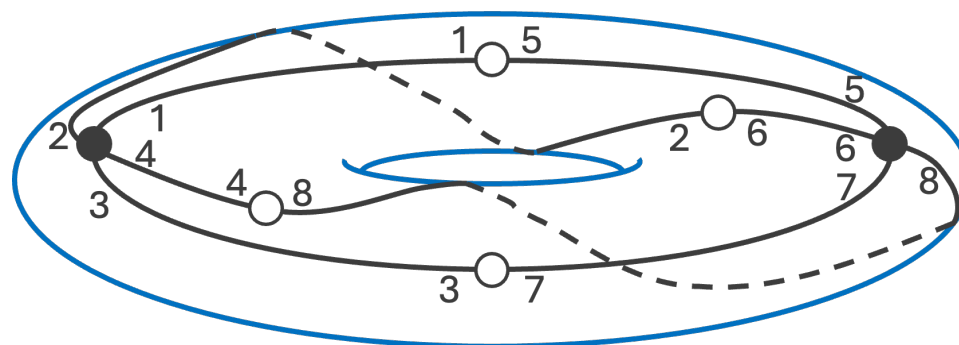
Nonregular



$$x = (1\ 2\ 3\ 4),\ y = (3\ 5\ 4)$$

$$G \cong S_5,\ \text{Aut } \mathcal{D} \cong \{1\}$$

Regular



$$x = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8),\ y = (1\ 5)(2\ 6)(3\ 7)(4\ 8)$$

$$G \cong \text{Aut } \mathcal{D} \cong C_4 \times C_2$$

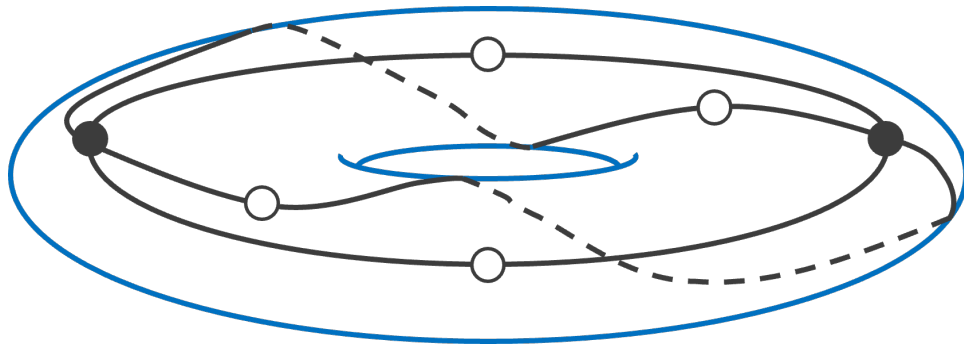
Uniformity and Regularity



Fact: A regular dessin $\xRightarrow{\quad} \not\Leftarrow$ a uniform dessin

Two dessins with the same uniform passport $[4^2, 2^4, 4^2]$

Regular

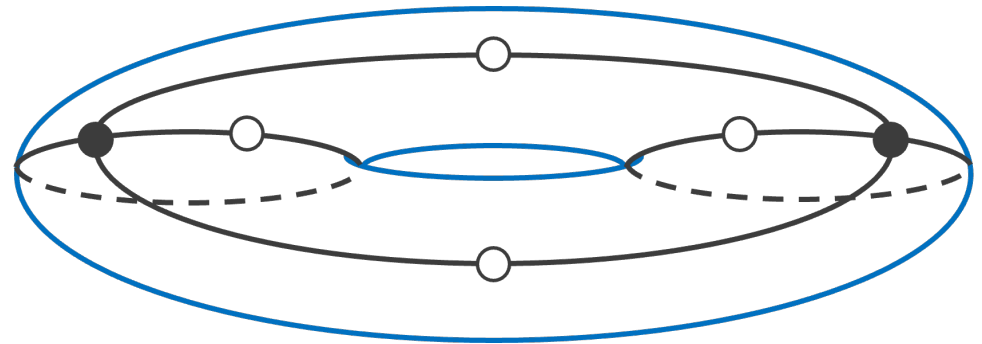


$$X: y^2 = x(x-1)(x+1)$$

$$\beta = -\frac{(x-1)^2(x+1)^2}{4x^2}$$

$$G \cong \text{Aut } \mathcal{D} \cong C_4 \times C_2$$

Nonregular



$$X: y^2 = x^3 - 11x - 14$$

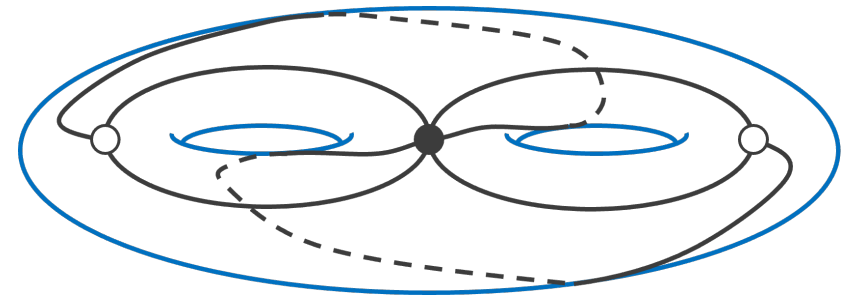
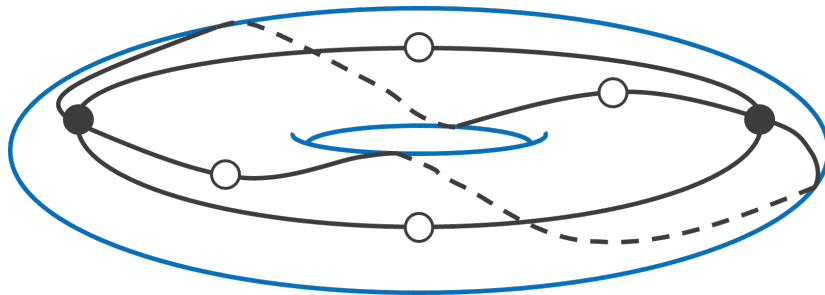
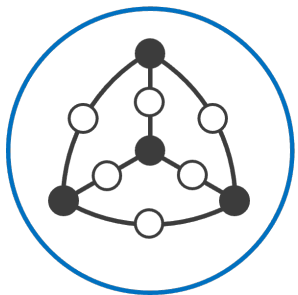
$$\beta = \frac{(x+1)^4}{16(x+2)^2}$$

$$G \cong (C_4 \times C_2) \rtimes C_2, \text{ Aut } \mathcal{D} \cong C_2 \times C_2$$

Main Problems

For a **uniform passport**:

- Under what conditions do **regular dessins d'enfants** exist, and how many are there?
- How are **automorphism groups** distributed?
- How does this distribution change with the **genus**?



Conjectural Picture of Automorphism Groups

Conjectural Distribution for **Uniform Passports**
(based on computational experiments)

- n : number of edges
(= degree of the Belyĭ func.)
- $p, q, r \geq 2$
- $\checkmark$: always occurs,
— : never occurs,
 $\diamond$: depends on the passport

Genus	Passport	Aut $\mathcal{D}$		
		1(trivial)	2 – n-1	n (reg.)
0	All	—	—	$\checkmark$
1	$[a^p, b^q, n]$	—	—	$\checkmark$
	$[a^p, b^q, c^r]$		$\checkmark$	$\diamond$
≥ 2	$[n, n, n]$	$\checkmark$	$\diamond$	$\checkmark$
	$[n, b^q, n]$		$\checkmark$	—
	$[b^q, b^q, n]$			$\diamond$
	$[a^p, b^q, n]$			$\diamond$
	$[a^p, b^q, c^r]$			$\diamond$

Main Results

Genus 0, 1

We proved the following results for uniform passports:

- (1) Genus 0: Every dessin is regular
- (2) Genus 1: The passport admits only regular dessins
 $\iff$ The passport contains a part n
- (3) Genus 1: No dessin has a trivial automorphism group
- (4) All genera: $[a^p, b^q, n]$ admits a regular dessin
 $\iff \gcd(p, q) = 1$

		Aut $\mathcal{D}$		
Genus	Passport	1(trivial)	2 – n-1	n (reg.)
0	All	—	— (1)	✓
1	$[a^p, b^q, n]$	—	— (2)	✓
	$[a^p, b^q, c^r]$	(3)	✓	◇ (4)

✓: always occurs, —: never occurs, ◇: depends on the passport

Genus ≥ 2

Genus ≥ 2

Original Conjecture

Genus	Passport	Aut $\mathcal{D}$		
		1(trivial)	2 – n-1	n (reg.)
≥ 2	$[n, n, n]$	✓	◇	✓
	$[n, b^q, n]$			
	$[b^q, b^q, n]$		✓	—
	$[a^p, b^q, n]$			◇
	$[a^p, b^q, c^r]$			◇

✓: always occurs, —: never occurs, ◇: depends on the passport

Genus ≥ 2

Current Picture

?: Still conjectural

Triviality

$|\text{Aut } \mathcal{D}|$

Every divisor of n

Genus	Passport	1(trivial)	2 – n-1	n (reg.)
≥ 2	$[n, n, n]$	✓	— (n : prime) ✓ (n : comp.)	✓
	$[n, b^q, n]$		✓?	
	$[b^q, b^q, n]$		✓ $\rightarrow$ $\diamond$	—
	$[a^p, b^q, n]$	✓?	✓?	✓ ($\gcd(p, q) = 1$) — ($\gcd(p, q) \neq 1$)
	$[a^p, b^q, c^r]$	✓ $\rightarrow$ $\diamond$	✓?	$\diamond$

✓: always occurs, —: never occurs, $\diamond$: depends on the passport

Counterexamples

Uniform Passports with Trivial $\text{Aut } \mathcal{D}$



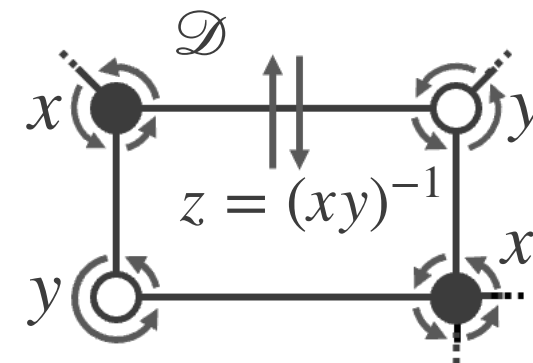
Theorem. [O. 2026]

Every uniform passport below with genus ≥ 2 admits a dessin d'enfant with trivial automorphism group.

(a) $[n, n, n]$

(b) $[n, b^q, n]$, $[b^q, n, n]$, and $[n, n, b^q]$ ($q \geq 2$)

(c) $[b^q, b^q, n]$, $[b^q, n, b^q]$, and $[n, b^q, b^q]$ ($q \geq 2$)



Approaches to the Proof

1. Construct explicit permutations x and y with $\text{Aut } \mathcal{D} \cong \{1\}$
or
2. Use estimates for the sizes of centralizers
or
3. Analyze the primitivity of the monodromy group

Construct explicit permutations x and y



Passport $[n, n, n]$ (genus $\geq 2 \rightarrow n$: odd, $n \geq 5$)

$$x = (1 \ 2 \ \dots \ n), \quad y = (2 \ 4 \ 6 \ \dots \ n-1 \ \underbrace{n \ n-2 \ \dots \ 3}_{\text{odd}} \ 1)$$



even

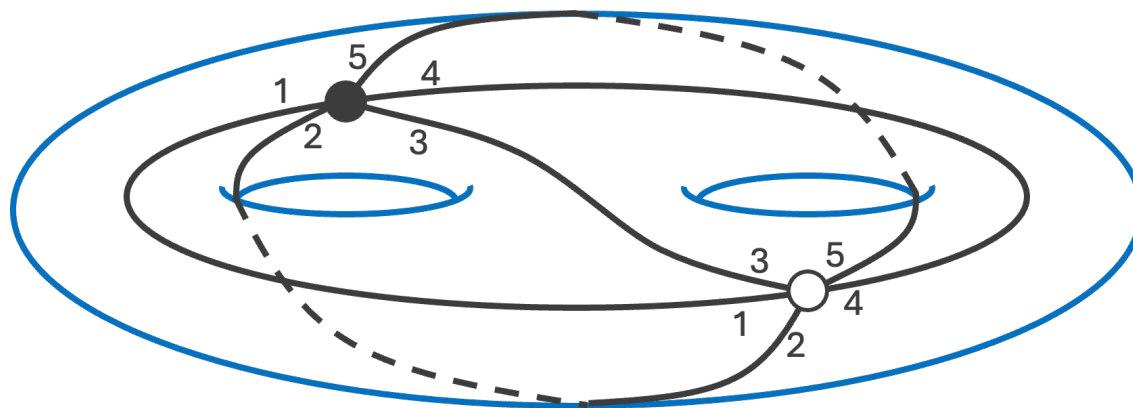
odd

- $(xy)^{-1} = (1 \ n-1 \ n \ n-3 \ n-2 \ \dots \ 4 \ 5 \ 2 \ 3)$

- $x^2(xyxy)^{-1} = (1 \ 2 \ 3) \rightarrow G = \langle x, (1 \ 2 \ 3) \rangle = A_n$

(Alternative group)

$$\rightarrow \text{Aut } \mathcal{D} \cong C_{S_n}(A_n) \cong \{1\}$$



$$[5, 5, 5]$$

$$x = (1 \ 2 \ 3 \ 4 \ 5)$$

$$y = (2 \ 4 \ 5 \ 3 \ 1)$$

$$\text{Aut } \mathcal{D} \cong \{1\}$$

Use estimates for the sizes of centralizers

Passport $[n, b^q, a^p]$ (genus ≥ 2)

$x \in S_n$: fixed n -cycle (e.g. $(1\ 2\ \dots\ n)$)

$$\begin{cases} T(b, q) := \{y \in S_n \mid y \text{ has cycle type } (b^q)\} \\ N(b, q, a) := \{y \in T(b, q) \mid (xy)^{-1} \text{ has cycle type } (a^p)\} \\ C(b, q) := \{y \in T(b, q) \mid \text{Aut } \mathcal{D} \not\cong \{1\}\} \\ D(n) := \{y \in S_n \mid \text{Aut } \mathcal{D} \not\cong \{1\}\} \end{cases}$$

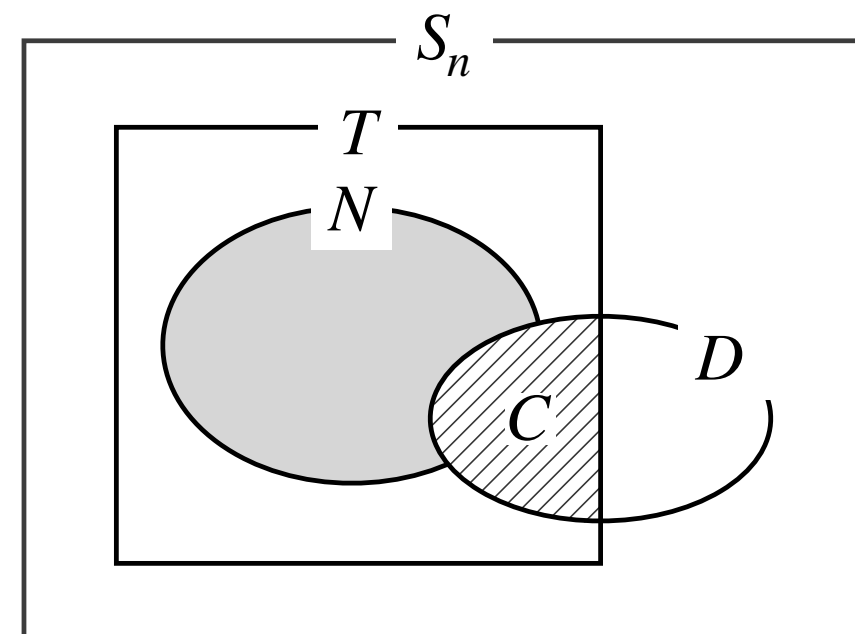
$[n, b^q, a^p]$ admits $\mathcal{D}$ s.t. $\text{Aut } \mathcal{D} \cong \{1\}$

$$\iff N \cap C^c \neq \emptyset \iff |N| > |D|$$

(1) Obtain lower bound of $|N|$

(2) Obtain upper bound of $|D|$

(3) Show $(1) > (2)$



Lower Bound of $|N(n, b, a)|$

$x \in S_n$: fixed n -cycle (e.g. $(1 \ 2 \ \dots \ n)$)

$T(b, q) := \{y \in S_n \mid y \text{ has cycle type } (b^q)\}$

$N(b, q, a) := \{y \in T(b, q) \mid (xy)^{-1} \text{ has cycle type } (a^p)\}$

Theorem. [Goupil and Schaeffer 1998]

Let $\lambda = (\lambda_1, \dots, \lambda_l)$, $\mu = (\mu_1, \dots, \mu_m)$: partitions of n , Passport $[\lambda, \mu, n]$
 $g = (n - (l + m) + 1)/2$, π : a fixed n -cycle in S_n .

Then **the number of solutions $(\sigma, \rho) \in C_\lambda \times C_\mu$ to $\sigma\mu = \pi$** is

$$c_{\lambda\mu}^n = \frac{n}{z_\lambda z_\mu 2^{2g}} \sum_{\substack{g_1, g_2 \geq 0 \\ g_1 + g_2 = g}} (l + 2g_1 - 1)!(m + 2g_2 - 1)! \sum_{\substack{(i_1, \dots, i_l) \models g_1 \\ (j_1, \dots, j_m) \models g_2}} \prod_{k=1}^l \binom{\lambda_k}{2i_k + 1} \prod_{k=1}^m \binom{\mu_k}{2j_k + 1},$$

where $z_\lambda = \prod_i \alpha_i! i^{\alpha_i}$ for a partition $\lambda = 1^{\alpha_1} \dots n^{\alpha_n}$.

Lower Bound of $|N(n, b, a)|$

Using the theorem by Goupil and Schaeffer,

Theorem. [O. 2026] Passport $[n, b^q, n]$

Let $n \geq 3$, $n = bq$, $b \geq 2$, and $n \equiv q \pmod{2}$. Then

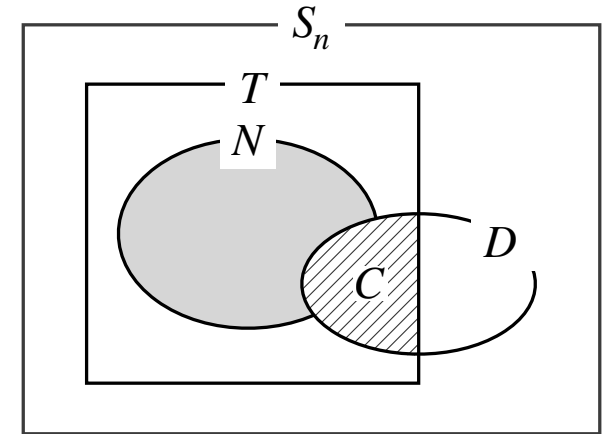
$$\frac{N(b, q, n)}{T(b, q)} \geq \frac{2}{n+2}.$$

The equality holds if and only if $b = 2$.

Proposition. [O. 2026] Passport $[n, b^q, b^q]$

For a passport $[n, b^q, b^q]$ with $n = qb$ and genus ≥ 1 ,

$$|N(b, q, b)| \geq \frac{(n-q)!}{2^{q(b-2)} \left(\frac{q+1}{2}\right)! \left(\frac{q-1}{2}\right)!} \left(\frac{b-1}{2b}\right)^{\frac{q-1}{2}}.$$



Upper Bound of $|D(n)|$

$x \in S_n$: fixed n -cycle (e.g. $(1 \ 2 \ \dots \ n)$)

$$D = D(n) := \{y \in S_n \mid \text{Aut } \mathcal{D} \not\cong \{1\}\}$$

Proposition.

For $D_k := C_{S_n}(x^k) = \{c \in S_n \mid cx^k = x^kc\}$ ($1 \leq k \leq n-1$),

$$(1) D_1 = \{x^k \mid 0 \leq k \leq n-1\}$$

$$(2) k \mid l \implies D_k \subset D_l \quad (1 \leq k, l \leq n-1)$$

(3) For $1 \leq k \leq n-1$ and $m = \gcd(n, k)$,

$$D_k = D_m, \quad |D_k| = |D_m| = \left(\frac{n}{m}\right)^m m!$$

$$(4) D = \bigcup_{k=1}^{n-1} D_k = \bigcup_{\ell: \text{ prime}, \ell \mid n} D_{\frac{n}{\ell}}$$

e.g.

$$n = 5: D = D_1$$

$$n = 8: D = D_4$$

$$n = 12: D = D_6 \cup D_4$$

Upper Bound of $|D(n)|$

Lemma.

$$\begin{aligned}
 |D| &\leq \sum_{\ell: \text{ prime, } \ell|n} \left| D_{\frac{n}{\ell}} \right| = \sum_{\ell: \text{ prime, } \ell|n} \ell^{\frac{n}{\ell}} \left(\frac{n}{\ell} \right)! \\
 &\leq \left| D_{\frac{n}{2}} \right| + \left| D_{\frac{n}{3}} \right| + \left| D_{\frac{n}{5}} \right| + \dots \leq \kappa_1 \cdot 2^{\frac{n}{2}} \left(\frac{n}{2} \right)!,
 \end{aligned}$$

$$n : \text{ odd} \implies |D| \leq \kappa_2 \cdot 3^{\frac{n}{3}} \left(\frac{n}{3} \right)!,$$

$$\text{where } \kappa_1 = \frac{2623}{1894} = 1.384\dots, \quad \kappa_2 = \frac{972}{947} = 1.026\dots$$

Use estimates for the sizes of centralizers

Passport $[n, b^q, b^q]$, $b \geq 5$ (genus $\geq 2 \rightarrow q$: odd, $q \geq 3$)

$$|N(b, q, b)| \geq N_{lower} := \frac{(n - q)!}{2^{q(b-2)} \left(\frac{q+1}{2}\right)! \left(\frac{q-1}{2}\right)!} \left(\frac{b-1}{2b}\right)^{\frac{q-1}{2}},$$

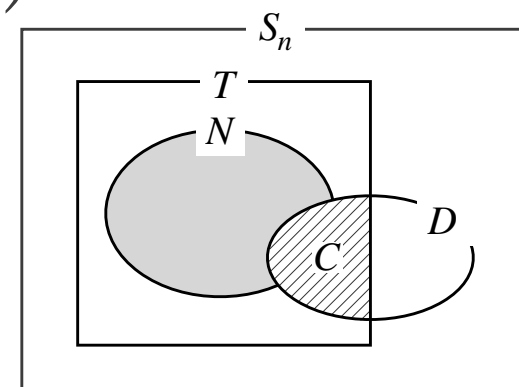
$$|D(n)| \leq D_{upper} := \kappa_2 \cdot 3^{\frac{n}{3}} \left(\frac{n}{3}\right)!.$$

Let $R(b, q) := N_{lower}/D_{upper}$. We can show

$$(1) R(5, 3) > 1, \quad (2) \frac{R(5, q+2)}{R(5, q)} > 1, \quad (3) \frac{R(b+2, q)}{R(b, q)} > 1.$$

$$\rightarrow |N(b, q, b)| > |D(n)|$$

※ For $b = 3$, the above argument does not apply
(in fact, $R(3, q) < 1$).



Construct explicit permutations x and y



Passport $[n, b^q, b^q]$, $b = 3$ (genus $\geq 2 \rightarrow q$: odd, $q \geq 3$)

$$x = (1 \ 2 \ \dots \ 3q), \ y = \alpha \beta_0 \beta_1 \cdots \beta_{\frac{q-5}{2}} \gamma,$$

$$\alpha = (1 \ 2 \ 4)(3 \ 7 \ 5),$$

$$\beta_i = \begin{cases} (11v+6 \ 11v+9 \ 11v+10)(11v+8 \ 3q-v \ 11v+13) & (i \equiv 0 \pmod{2}) \\ (11v \ 11v+1 \ 11v+4)(11v+3 \ 11v+7 \ 11v+5) & (i \equiv 1 \pmod{2}) \end{cases},$$

$$\gamma = \begin{cases} (11w+11 \ 11w+12 \ 11w+14) & (q \equiv 1 \pmod{4}) \\ (11w+6 \ 11w+8 \ 11w+9) & (q \equiv 3 \pmod{4}) \end{cases},$$

where $v = \lfloor i/2 \rfloor$, $w = \lfloor (q-3)/4 \rfloor$

$$q = 3: y = (1 \ 2 \ 4)(3 \ 7 \ 5)(6 \ 8 \ 9)$$

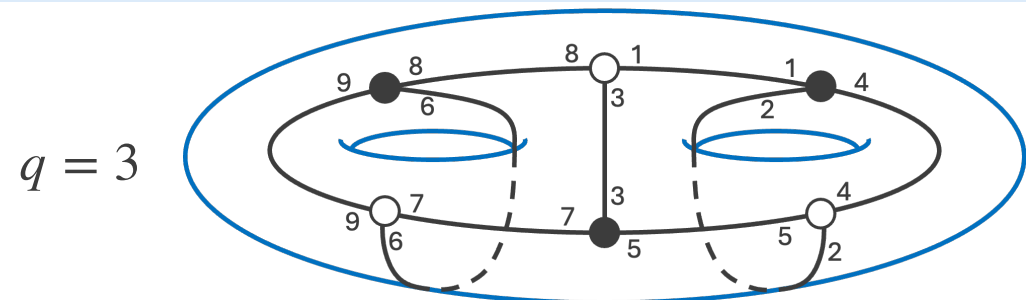
$$q = 5: y = (1 \ 2 \ 4)(3 \ 7 \ 5)(6 \ 9 \ 10)(8 \ 15 \ 13)(11 \ 12 \ 14)$$

(1) y has cycle type (3^q)

(2) $(xy)^{-1}$ has cycle type (3^q)

(3) $\text{Aut } \mathcal{D} \cong \{1\}$

Every $[n, b^q, b^q]$ with genus ≥ 2 admits $\mathcal{D}$ with $|\text{Aut } \mathcal{D}| = 1$



Other Results

Passport $[n, n, n]$

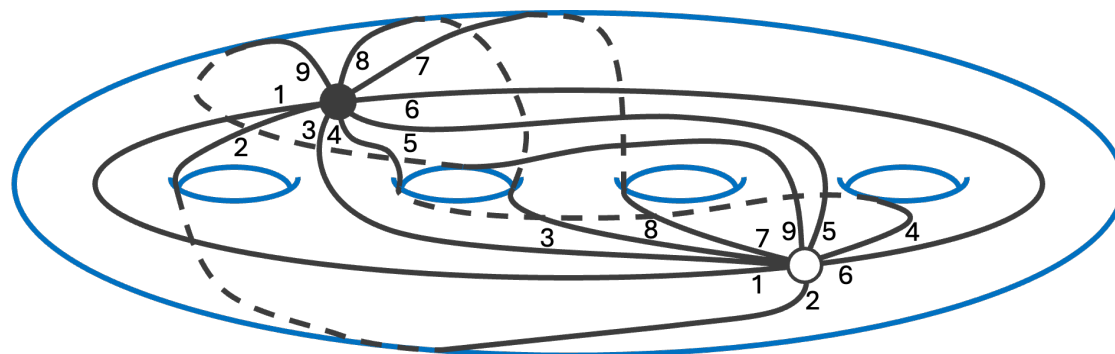
$\downarrow$ Genus	$ \text{Aut } \mathcal{D} \rightarrow$	1(trivial)	$2 - n-1$	n (reg.)
$[n, n, n]$	✓	— (n : prime) ✓ (n : comp.)	✓	✓

Theorem. [O. 2026]

For the passport $[n, n, n]$ with genus ≥ 2 ,

- It admits a nonregular dessin with nontrivial $\text{Aut } \mathcal{D}$
 $\iff n$ is composite
- For every r such that $1 < r < n$ and $r \mid n$, there exists a dessin $\mathcal{D}$ such that $|\text{Aut } \mathcal{D}| = r$.

$[9, 9, 9], |\text{Aut } \mathcal{D}| = 3$



Only Dessins with $\text{Aut } \mathcal{D} \cong \{1\}$



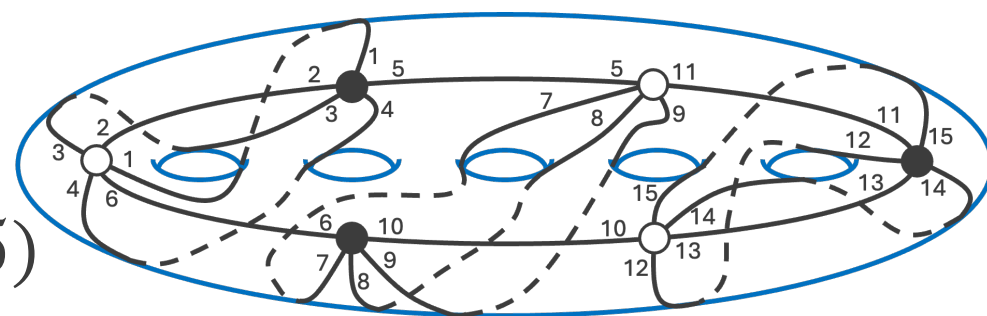
↓Genus	$ \text{Aut } \mathcal{D} \rightarrow$	1(trivial)	2 – n-1	n (reg.)
$[b^q, b^q, n] \ (q \geq 2)$	✓	✓ → ◇	—	

Passport $[15, 5^3, 5^3]$, genus 5

- With x of cycle type (15) fixed, there are 4354560 permutations y
- Up to conjugation by $C_{S_n}(x)$, these form 290304 conjugacy classes
- $|\text{Aut } \mathcal{D}| = 1$ for all y . i.e.,

Only dessins with $\text{Aut } \mathcal{D} \cong \{1\}$

(The only such passport for $n \leq 15$)



$[5^3, 5^3, 15], \text{Aut } \mathcal{D} \cong \{1\}$

Theorem. [O. 2026]

ℓ_1, ℓ_2 : distinct odd primes, $\ell_1 < \ell_2$

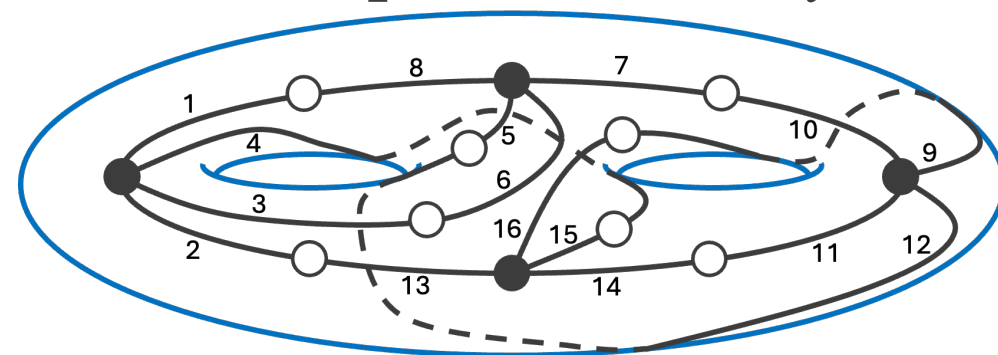
For every dessin $\mathcal{D}$ with passport $[\ell_2^{\ell_1}, \ell_2^{\ell_1}, \ell_1 \ell_2], \text{Aut } \mathcal{D} \cong \{1\}$.

No Dessins with $\text{Aut } \mathcal{D} \cong \{1\}$

$\downarrow$ Genus	$ \text{Aut } \mathcal{D} \rightarrow$	1(trivial)	2 – n-1	n (reg.)
$[a^p, b^q, c^r]$ ($p, q, r \geq 2$)	✓ $\rightarrow \diamond$		✓*	$\diamond$

Passport $[8^2, 2^8, 4^4]$, genus 2

- With x of cycle type (8^2) fixed, there are 920 permutations y
- Up to conjugation by $C_{S_n}(x)$, these form 19 conjugacy classes
- $|\text{Aut } \mathcal{D}| = 2, 4, 8, 16$ i.e.,



No dessins with $\text{Aut } \mathcal{D} \cong \{1\}$ $[4^4, 2^8, 8^2]$, $|\text{Aut } \mathcal{D}| = 16$ (regular)

(The only such passport for $n \leq 16$, genus ≥ 2)

$[(4t)^2, 2^{4t}, (2t)^4]$ ($t \geq 2$) has no dessins with $|\text{Aut } \mathcal{D}| = 1$?

$\rightarrow$ False! (Already fails for $t = 3$)

Open Problems

- Resolve the remaining conjectural cases
 - ♦ Existence of dessins with $\text{Aut } \mathcal{D} \cong \{1\}$ for $[a^p, b^q, n]$,
 - ♦ When does $[a^p, b^q, c^r]$ admit such a dessin?
- Quantitative study of the distribution of regular dessins and automorphism groups

Genus	Passport	Aut $\mathcal{D}$		
		1(trivial)	2 – n-1	n (reg.)
≥ 2	$[n, n, n]$	✓	— (n: prime) ✓ (n: comp.)	✓
	$[n, b^q, n]$		✓?	
	$[b^q, b^q, n]$		✓ → ◇	—
	$[a^p, b^q, n]$	✓?	✓?	✓ (gcd(p, q) = 1) — (gcd(p, q) ≠ 1)
	$[a^p, b^q, c^r]$	✓ → ◇	✓?	◇

✓: always occurs, —: never occurs, ◇: depends on the passport

Thank you for your attention